UiO : Department of Mathematics
University of Oslo

Hermitian K-theory of Finite Fields via the Motivic Adams Spectral Sequence

Jonas Irgens Kylling

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Abstract

We work in the stable motivic homotopy category over finite fields of odd characteristic. Here we use the motivic Adams spectral sequence to compute the 2-completed motivic homotopy groups of a spectrum representing Hermitian K-theory. As a corollary we obtain the 2-completed Hermitian K-groups of finite fields of odd characteristic. Our results agree with earlier results by Friedlander [Fri76].

Contents

Introduction

Hermitian K-theory of finite fields classifies vector spaces with symmetric bilinear forms. More generally, hermitian K-theory of schemes classifies vector bundles with symmetric bilinear forms. Hermitian K-theory plays the same role for schemes, as real K-theory plays for topological spaces. The Hermitian K-theory of finite fields was first computed by Friedlander in [Fri76]. The reduced groups are

m modulo 8	0	1	2	3	4	5	6	7
$\widetilde{KO}_m(\mathbb{F}_q)$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$\dfrac{\mathbb{Z}}{q^{(m+1)/2}-1}$	0	0	0	$\dfrac{\mathbb{Z}}{q^{(m+1)/2}-1}$

It is well known that real K-theory of topological spaces is represented by an Ω-spectrum KO. That is, we have isomorphisms

$$KO_iX = [X, KO]_i.$$

Computing the stable homotopy classes of maps from X to BO is a hard problem, but some tools are available, for instance the Adams spectral sequence. For a prime number p the Adams spectral sequence is a bigraded spectral sequence with E_2-page

$$E_2 = \mathrm{Ext}_{A_p}(H_*(X,\mathbb{Z}/p), H_*(ko,\mathbb{Z}/p)),$$

which is strongly convergent to $[X, ko]_{*,\hat{p}}$, the p-completed groups. Here ko is the connective cover of KO and A_p is the dual of the Steenrod algebra. Of particular interest is the sphere spectrum S. It is the unit object in the stable homotopy category and a building block for other spectra. In this case the Adams spectral sequence is strongly convergent to $\pi_*(ko)_{\hat{p}}$.

Motivic homotopy theory

With the advent of motivic homotopy theory these topological techniques became available for smooth schemes over a field, Sm/k. Voevodsky and others constructed a stable homotopy category which contained Sm/k. The motivic homotopy category has many similarities to the ordinary homotopy category for topological spaces. It is triangulated and symmetric monoidal, we have motivic cohomology, motivic homotopy groups, a motivic Steenrod algebra and there exists a motivic Adams spectral sequence. These are directly related to classical objects of study in algebraic geometry. Hence, the algebraic topologists toolset is available for obtaining reults in algebraic geometry. Of particular interest to us, is the existence of a spectrum **KO** representing Hermitian K-theory of schemes. If we let the base field be a finite field $k = \mathbb{F}_q$, Hermitian K-theory of finite fields can be read directly off from the motivic homotopy groups of **KO**. That is, $\pi_m\mathbf{KO} = KO_m(\mathrm{Spec}\,\mathbb{F}_q)$.

Calculation of $\pi_\star\mathbf{KO}$

The aim of this thesis is to calculate the motivic homotopy groups of **KO** with the motivic Adams spectral sequence over $k = \mathbb{F}_q$, where q is odd. This calculation is in the same spirit as the computation of Ormsby in [Orm11], but he works over p-adic fields and uses a motivic version of the Brown-Peterson spectrum in place of **KO**. Knowledge of the homotopy groups give us the hermitian K-theory of finite fields. We will be calculating at the prime 2, hence we only obtain the 2-completed motivic homotopy groups. In this case the E_2-page of the motivic Adams spectral sequence is

$$E_2 = \mathrm{Ext}_{A_\star}(H_\star(S;\mathbb{Z}/2), H_\star(ko;\mathbb{Z}/2)).$$

Here ko is a kind of connective cover of **KO**, which should satisfy some finiteness conditions. If this is the case the spectral sequence is strongly convergent to $\pi_\star(ko)_{\hat{2}}$. To get the E_2-page we need to know $A_\star$, the dual of the motivic Steenrod algebra mod 2, and the motivic homology groups $H_\star(S;\mathbb{Z}/2)$, $H_\star(ko;\mathbb{Z}/2)$. To compute the Ext-group we reduce to the case of an algebraically closed base field and use a Bockstein spectral sequence to return to the Ext-group we need. As for **KO**, a motivic Adams spectral sequence can be set up for $H\mathbb{Z}_{(2)}$, the spectrum representing motivic cohomology with $\mathbb{Z}_{(2)}$ coefficients. In this case the abutment is $H_\star(\operatorname{Spec}\mathbb{F}_q;\mathbb{Z}_2)$. These groups can be computed in terms of the corresponding groups in étale cohomology. Since the abutment is known this determines all the differentials in the spectral sequence for $H\mathbb{Z}_{(2)}$. The algebra structure and a map $\mathbf{KO} \to H\mathbb{Z}_{(2)}$, then determines all the differentials in the spectral sequence for **KO**. Then it is straightforward to calculate the E_∞-page and the abutment. The result agrees with the result of Friedlander [Fri76]. The model we use for the connective cover of **KO** is the zeroth effective functor $f_0\mathbf{KO}$. This model have several good properties, but we do not know whether it satisfies the required finiteness properties. Hence, we do not have strong convergence. Modulo this missing detail our argument is complete.

Outline of the thesis

The outline of the thesis is as follows: In Chapter 1 we introduce motivic homotopy theory, motivic chomology, the motivic Steenrod algebra, some motivic spectra and properties of spectra. In the last section we calculate the motivic homology groups which are the input to the E_2-page of the motivic Adams spectral sequence for ko. In Chapter 2 we discuss Milnor K-theory of fields, in particular finite fields, and sketch the construction of Hermitian K-theory and the higher K-theory groups. Next, in Chapter 3 we introduce spectral sequences and define the main spectral sequences we make use of: The Bockstein spectral sequence, the motivic Adams spectral sequence and the slice spectral sequence. In the final chapter we calculate the motivic homotopy groups of **KO**. At the end are two appendices. Appendix A states various results on Hopf algebroids, Ext of comodules and the cobar complex. Most of the results can be found in [Rav86, Appendix A]. Appendix B contains some miscellaneous results from elementary number theory.

The first three sections of Chapter 1 should make the reader well acquainted with the basics of motivic homotopy theory. A knowledge of algebraic topology is an advantage. Section 1.8 and parts of Section 1.5 are fairly technical and can be skipped by the reader not seeking all the details. The first section of Chapter 2 should be readable for anyone with some knowledge of algebra. Section 2.2 and Section 2.3 are a bit brief. The reader may consult consult [Bak81] or [Wei13] for further details. After having read Chapter 1 and Chapter 3 for the necessary background, Chapter 4 should be readable. However, experience calculating with spectral sequences is undoubtedly an advantage. Appendix A might be skipped by the expert, but fills in some of the details of [Rav86, Appendix A]. Appendix B is accessible for anyone with an introductory course in algebra.

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1 Motivic Homotopy Theory

1.1 Construction

In this section we sketch the construction of the motivic homotopy category and the motivic stable homotopy category. We follow the presentation of [Jar00] and [Voe98].

We start with the category of smooth separated schemes of finite type over k, Sm/k, which is embedded into a larger category with better categorial properties. This category is denoted by $\mathrm{Spc}(k)$, the motivic spaces. Next, $\mathrm{Spc}(k)$ is provided with a model structure which is localized with respect to projections $X \times \mathbb{A}^1 \to X$. The homotopy category of this model structure is the motivic homotopy category $\mathcal{H}(k)$. To obtain the stable homotopy category we first construct a category of spectra $\mathrm{Spt}(k)$ from $\mathrm{Spc}_\bullet(k)$ and provide it with a model structure. The homotopy category of $\mathrm{Spt}(k)$ is the motivic stable homotopy category, $\mathcal{SH}(k)$. We will now elaborate on this, but model categories and abstract homotopy theory is not the main focus of this thesis, so we will be brief. For complete expositions we refer the reader to [Jar00] or [Voe98]. They give different, but equivalent, constructions of the motivic unstable and stable homotopy categories.

Let Sm/k be the category of smooth schemes over k. This category does not have all colimits, which is necessary to apply abstract homotopy theory. Therefore we consider the category of simplicial presheaves, $\Delta^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}/k)$. The symbols are explained below. Here $\mathrm{Pre}(\mathrm{Sm}/k)$ is the category of presheaves on Sm/k. Objects are functors $(\mathrm{Sm}/k)^{\mathrm{op}} \to \mathrm{Set}$, and morphism are natural transformations. Let Δ be the simplicial category (e.g., [Wei94, Section 8.1]). The objects are ordered sets $[n] := \{0 < 1 < \cdots < n\}$, and the morphisms are monotone nondecreasing functions. If $\mathcal{C}$ is a category we denote by $\Delta^{\mathrm{op}}\mathcal{C}$ the category of simplicial objects in $\mathcal{C}$. This is the category with objects functors $\Delta^{\mathrm{op}} \to \mathcal{C}$ and morphisms natural transformations. The category of simplicial presheaves $\Delta^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}/k)$ is both complete and cocomplete, hence, suitable for abstract homotopy theory. There is an embedding of Sm/k into $\mathrm{Pre}(\mathrm{Sm}/k)$ via the Yoneda embedding $X \mapsto \mathrm{Hom}(-, X)$. There is a further embedding of $\mathrm{Pre}(\mathrm{Sm}/k)$ into $\Delta^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}/k)$, mapping a presheaf X to the constant simplicial presheaf $[n] \mapsto X$. Similarly, there is an embedding of simplicial sets $\Delta^{\mathrm{op}}\mathrm{Set} \to \Delta^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}/k)$, mapping a simplicial set X to the simplicial presheaf taking the value X on all objects of Sm/k. Via these embeddings we will by abuse of notation use the same symbols for objects in Sm/k (respectively $\Delta^{\mathrm{op}}\mathrm{Set}$) and their image in $\Delta^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}/k)$. With such nice properties this category deserves to be the category of k-spaces, $\mathrm{Spc}(k)$, or simply motivic spaces

$$\mathrm{Spc}(k) := \Delta^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}/k).$$

Note that $\operatorname{Spec} k$ is the terminal object of $\mathrm{Spc}(k)$.

We then give $\mathrm{Spc}(k)$ a model structure (e.g., [Hov99, Ch. 1]). This is the local model structure that takes into account the Nisnevich topology of Sm/k. The local model structure is then modified (or localized with respect to $* \to \mathbb{A}^1$) to give the motivic model structure. This model structure is proper and closed [Voe98, Theorem 3.7]. The associated homotopy category is the motivic homotopy category, $\mathcal{H}(k)$ (also know as the unstable motivic homotopy category). For further details see [Jar00, 1.1]. In accordance with standard notation in topology we denote $\mathrm{Hom}_{\mathcal{H}(k)}(X, Y)$ by $[X, Y]$.

Similarly, there is a pointed category $\mathrm{Spc}_\bullet(k)$ (i.e., the category with objects $\operatorname{Spec} k \to X$ and compatible morphisms). An analogous construction can be carried out for the pointed category and lead to the pointed motivic homotopy category $\mathcal{H}_\bullet(k)$. There is a canonical functor from an unpointed category to a pointed category given by $X \mapsto (X \coprod *, *) =: X_+$.

The category $\mathrm{Spc}_\bullet(k)$ has all quotients. Hence, several constructions from algebraic topology are now available in $\mathrm{Spc}_\bullet(k)$. A quotient of X by Y in $\mathrm{Spc}_\bullet(k)$ is the usual push-out square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ * \arrow[r] & X/Y \end{tikzcd}$$

and is pointed by Y. For a collection of pointed spaces $(X_i, x_i)_{i\in I}$, the wedge product $\bigvee_{i\in I}(X_i, x_i)$ is the usual push-out square

$$\begin{tikzcd} \coprod_{i\in I} x_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \coprod_{i\in I} X_i \arrow[d] \\ * \arrow[r] & \bigvee_{i\in I} X_i \end{tikzcd}$$

The wedge product is the coproduct in $\mathrm{Spc}_\bullet(k)$. Similarly, the smash product is the push-out square

$$\begin{tikzcd} X \vee Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \times Y \arrow[d] \\ * \arrow[r] & X \wedge Y \end{tikzcd}$$

Via the embeddings of Sm/k and $\Delta^{\mathrm{op}}\mathrm{Set}$ into $\mathrm{Spc}(k)$ we get two kinds of spheres. The simplicial sphere

$$S^1 := \Delta^1/\partial\Delta^1,$$

pointed by Δ^0. The Tate-circle (also known as the "geometric" circle or the "twisted" circle)

$$S^\alpha := \mathbb{A}^1 - 0,$$

pointed by 1. Another common notation for the Tate-circle $\mathbb{A}^1 - 0$ is $\mathbb{G}_m$. Arbitrary spheres are then formed by taking smash products. The $(m + n\alpha)$-sphere is

$$S^{m+n\alpha} := (S^1)^{\wedge m} \wedge (S^\alpha)^{\wedge n}. \tag{1.1}$$

This gives us suspension functors $\Sigma^{m+n\alpha}X := S^{m+n\alpha} \wedge X$. Below we will see that in the stable homotopy category, Σ^1 corresponds to a shift functor in a triangulated category, and it will simply be denoted by Σ. Note that the α is just a "basis vector", similar to the notation $1, i, j$ as a basis for $\mathbb{R}^3$. This grading convention is inspired by equivariant homotopy theory. When we introduce the stable motivic homotopy category below it will be possible to take desuspensions, and we get the sphere $S^{-1+\alpha}$. Another common grading convention is to denote this sphere by $S^{0,1}$, and the simplicial sphere by $S^{1,0}$. To translate between the different grading conventions use the transformations $(p, q) \mapsto p - q + q\alpha$, and $m + n\alpha \mapsto (m + n, n)$. With this convention $S^1_t = S^{1,1}$ and $\mathbb{P}^1 \simeq S^{2,1}$. We prefer to use the $(m + n\alpha)$-grading when we work with motivic homotopy. When we discuss motivic cohomology in Section 1.2 we use the (p, q)-grading.

One useful construction on vector bundles in algebraic topology is the formation of Thom-spaces. This is also available in $\mathrm{Spc}_\bullet(k)$.

Definition 1.1.1. Consider a vector bundle $\mathcal{E} \to X$ in Sm/k with zero section $s : X \to \mathcal{E}$. We construct the Thom-space $\mathrm{Th}(\mathcal{E} \to X) := \mathcal{E}/(\mathcal{E} - s(X))$ in $\mathrm{Spc}_\bullet(k)$.

The following lemma summarize the properties of Thom-spaces which are important to us. In Section 1.3 we use it to construct the motivic cobordism spectrum, **MGL**.

Lemma 1.1.2 ([Dun+07, Example 2.25])**.** *Let $\mathcal{E} \to X$, $\mathcal{E}_1 \to X_1$ and $\mathcal{E}_2 \to X_2$ be vector bundles. Then we have the following equalities in $\mathcal{H}_\bullet(k)$:*

1. $\mathrm{Th}(\mathcal{E}_1 \times \mathcal{E}_2 \to X_1 \times X_2) = \mathrm{Th}(\mathcal{E}_1 \to X_1) \wedge \mathrm{Th}(\mathcal{E}_2 \to X_2)$.
2. $\mathrm{Th}(\mathbb{A}^1 \times X \to X) = \mathbb{A}^1/(\mathbb{A}^1 - 0)$.

Proposition 1.1.3 ([Voe98, Lemma 4.1])**.** *In the pointed homotopy category we have canonical isomorphisms*

$$(\mathbb{A}^n - 0, 1) \cong S^{n(1+\alpha)-\alpha},$$
$$\mathbb{P}^n/\mathbb{P}^{n-1} \cong \mathbb{A}^n/(\mathbb{A}^n - 0) \cong S^{n(1+\alpha)}.$$

In particular $\mathbb{P}^1 \cong \mathbb{A}^1/(\mathbb{A}^1 - 0) \cong S^{1+\alpha}$.

Proof. We only prove the second claim when $n = 1$. The rest of the proof is found in [MV99, pp. 110-113].

Consider the two push-out squares

$$\begin{tikzcd} \mathbb{A}^1 - \{0\} \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner"] & \mathbb{A}^1 \arrow[r, "\simeq"] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner"] & * \arrow[d] \\ \mathbb{A}^1 \arrow[r, tail] & \mathbb{P}^1 \arrow[r, "\simeq"] & \mathbb{A}^1/\mathbb{A}^1 - \{0\} \end{tikzcd}$$

The first square is a push-out square by gluing. The second is a push-out square by definition, and equals the push out of the entire rectangle. In the model structure, monomorphisms are cofibrations. Since cofibrations are preserved under push-outs we get the cofibrations in the diagram. The weak equivalence on the top row is due to $\mathbb{A}^1 \simeq *$, while the weak equivalence on the bottom row is a consequence of the model structure being proper. □

The category $\mathcal{H}_\bullet(k)$ is not a triangulated category, but we still have the notion of cofiber sequences, and distigiushed triangles. The motivic stable homotopy category is a triangulated category, and the distinguished triangles in $\mathcal{H}_\bullet(k)$ remain distinguished when transported to the stable homotopy category. The distinguished triangles are defined as sequences

$$A \to B \to C \to \Sigma A,$$

which are isomorphic in $\mathcal{H}_\bullet(k)$ to the cofiber sequences defined below. As the notation indicate, the shift functor is Σ^1.

Definition 1.1.4. Let $X \to Y$ be a morphism of pointed spaces. Then $\mathrm{Cone}(f)$ is the push-out square

$$\begin{tikzcd} X \arrow[r, "X\wedge 0"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & \Sigma X \arrow[d] \\ Y \arrow[r, "\eta_f"] & \mathrm{Cone}(f) \end{tikzcd}$$

A cofiber sequence is the induced sequence

$$X \xrightarrow{f} Y \xrightarrow{\eta_f} \mathrm{Cone}(f) \to \Sigma X$$

(the map $\mathrm{Cone}(f) \to \Sigma X$ is induced by the map $\mathrm{Cone}(f) \to \mathrm{Cone}(X \to *) \cong \Sigma X$).

We now proceed to construct the motivic stable homotopy category.

Definition 1.1.5. A k-spectrum is a sequence of pointed spaces $X := (X_n)_{n\in\mathbb{N}}, X_n \in \mathrm{Spc}_\bullet$ and structure maps $S^{1+\alpha} \wedge X_n \to X_{n+1}$. A map of spectra is a sequence of maps

$$(f : X \to Y) = (f_n : X_n \to Y_n)_{n\in\mathbb{N}}$$

commuting with the structure maps, i.e., $f_{n+1} = \sigma_Y(S^{1+\alpha} \wedge f_n)$. This category is denoted by $\mathrm{Spt}(k)$.

The coproduct in $\mathrm{Spt}(k)$ is given by $\oplus_\alpha E_\alpha = (\vee_\alpha E_{i,\alpha}, (\delta_i \circ (\vee e_{i,\alpha})))$, where δ_i is the canonical isomorphism $S^{1+\alpha} \wedge (\vee_\alpha E_{i,\alpha}) \to \vee_\alpha (S^{1+\alpha} \wedge E_{i,\alpha})$. The functor $X \mapsto (S^{n(1+\alpha)} \wedge X, \mathrm{Id})$ defines a stabilization functor $\Sigma^\infty : \mathrm{Spc}_\bullet(k) \to \mathrm{Spt}(k)$. By abuse of notation we denote $\Sigma^\infty X$ by X. This provides us with a functor $\mathrm{Spc}(k) \to \mathrm{Spt}(k)$ via the canonical map $\mathrm{Spc}(k) \to \mathrm{Spc}_\bullet(k)$. In particular a scheme $X \in \mathrm{Sm}/k$ gives rise to the spectrum $\Sigma^\infty X_+$. The sphere spectrum is defined to be $S := \Sigma^\infty \operatorname{Spec} k_+$. It is the zero object of $\mathrm{Spt}(k)$.

Remark 1.1.6. There is a more general construction of T-spectra for any compact object $T \in \mathrm{Spc}(k)$ (see Definition 1.1.14 below), such that T-spectra can be equipped with a model structure. In this language k-spectra are $S^{1+\alpha}$-spectra (or $\mathbb{P}^1$-spectra). Furthermore, if the symmetric group on 3 symbols acts trivially on $T \wedge T \wedge T$, the associated homotopy category is symmetric monoidal with respect to the smash product ([Jar00], [Voe98, Theorem 5.6]), see Proposition 1.1.9.

Characteristic of motivic homotopy theory are the bigraded homotopy groups.

Definition 1.1.7. For $m, n \in \mathbb{Z}, X \in \mathrm{Spt}(k)$ and $U \in \mathrm{Sm}/k$, consider the directed system

$$[S^{m+n\alpha} \wedge U_+, X_0] \to [S^{m+n\alpha+(1+\alpha)} \wedge U_+, X_1] \to [S^{m+n\alpha+2(1+\alpha)} \wedge U_+, X_2] \to \cdots$$

We define $\underline{\pi}_{m+n\alpha} X(U)$ to be the colimit of the system above. This defines a presheaf of stable motivic homotopy groups by $\underline{\pi}_{m+n\alpha} X : U \mapsto \underline{\pi}_{m+n\alpha} X(U)$. We denote the stable homotopy groups of X by $\pi_{m+n\alpha} X := \underline{\pi}_{m+n\alpha}(\operatorname{Spec} k)$. They are by definition the colimit of the system

$$[S^{m+n\alpha}, X_0] \to [S^{m+n\alpha+(1+\alpha)}, X_1] \to [S^{m+n\alpha+2(1+\alpha)}, X_2] \to \cdots,$$

very similar to the stable homotopy groups in topology.

The construction of the motivic model structure on $\mathrm{Spt}(k)$ is a much deeper dive into model structures and abstract homotopy theory than we wish to make. We refer the reader to [Jar00], but here is a quick sketch: The spectra $\mathrm{Spt}(k)$ inherits a levelwise model structure from $\mathrm{Spc}_\bullet(k)$. That is, a weak equivalence is a motivic weak equivalence on the constituent spaces of the spectra, and similarly for fibrations, while cofibrations are defined via the left lifting property [Hov99, Lemma 1.1.10]. Then we introduce a stabilization functor and a fibrant replacement functor, and use this to define a new model category structure, see [Jar00, 2.3].

The following lemma characterizes the stable equivalences in the model structure on $\mathrm{Spt}(k)$ in terms of the motivic homotopy groups.

Lemma 1.1.8 ([Jar00, Lemma 3.7])**.** *A map $X \to Y$ in $\mathrm{Spt}(k)$ is a stable equivalence if and only if it induces an isomorphism of presheaves of abelian groups $\underline{\pi}_{m+n\alpha} X \to \underline{\pi}_{m+n\alpha} Y$.*

The associated homotopy category is the motivic stable homotopy category $\mathcal{SH}(k)$. As always there is a localization functor $\mathrm{Spt}(k) \to \mathcal{SH}(k)$. By abuse of notation we will denote the image of X in $\mathcal{SH}(k)$ by X. The functor $\Sigma^\infty : \mathrm{Spc}_\bullet(k) \to \mathrm{Spt}(k)$ extends to a functor $\mathcal{H}_\bullet(k) \to \mathcal{SH}(k)$. All the categories we have considered so far fit nicely in a commutative diagram

$$\begin{tikzcd}
\mathrm{Sm}/k \arrow[r, hook] \arrow[d, "+"] & \mathrm{Spc}(k) \arrow[r, two heads] \arrow[d, "+"] & \mathcal{H}(k) \arrow[d, "+"] \\
\mathrm{Sm}/k_\bullet \arrow[r, hook] & \mathrm{Spc}_\bullet(k) \arrow[r, two heads] \arrow[d, "\Sigma^\infty"] & \mathcal{H}_\bullet(k) \arrow[d, "\Sigma^\infty"] \\
\Delta^{\mathrm{op}}\mathrm{Set}_\bullet \arrow[ur] & \mathrm{Spt}(k) \arrow[r, two heads] & \mathcal{SH}(k)
\end{tikzcd}$$

Distinguished triangles in $\mathcal{SH}(k)$ are defined similarly as in $\mathcal{H}_\bullet(k)$, Definition 1.1.4. The induced functor $\Sigma^\infty : \mathcal{H}_\bullet(k) \to \mathcal{SH}(k)$ preserves cofiber sequences and commutes with Σ^1-suspension. The suspension functors in Section 1.1 carry over to $\mathcal{SH}(k)$. For spectra X and Y denote $\mathrm{Hom}_{\mathcal{SH}(k)}$ by $[X, Y]$. We define the graded Hom-groups as

$$[X, Y]_{m+n\alpha} := [S^{m+n\alpha} X, Y].$$

In this notation $\pi_{m+n\alpha} E = [S, E]_{m+n\alpha}$.

Proposition 1.1.9 ([Voe98, 5.6])**.** *Here is a summary of the main properties of $\mathcal{SH}(k)$.*

- *$\mathcal{SH}(k)$ is an additive category.*
- *$\mathcal{SH}(k)$ is a triangulated category. The shift functor is Σ^1* [Voe98, Proposition 5.4].
- *$\mathcal{SH}(k)$ is a closed symmetric monoidal category with respect to the smash product.*
- *The sphere spectrum S is the unit with respect to the smash product.*
- *The smash product commutes with* hocolim.
- *For a spectrum E and a pointed space X, there is a canonical isomorphism between $E \wedge \Sigma X$ and $(E_i \wedge X, e_i \wedge X)$.*
- *There is a canonical isomorphism $(\oplus_i E_i) \wedge F \to \oplus_i (E_i \wedge F)$.*
- *The smash product preserves cofiber sequences. That is, for a cofiber sequence*

$$E \xrightarrow{f} F \to \mathrm{Cone}(f) \xrightarrow{\epsilon} \Sigma^1 E$$

  and a spectrum G, the sequence

$$E \wedge G \to F \wedge G \to \mathrm{Cone}(f) \wedge G \to \Sigma^1(E \wedge G)$$

  is a cofiber sequence. Here the last map is the composition of $\epsilon \wedge G$ with the canonical isomorphism $(\Sigma^1 E) \wedge G \to \Sigma^1(E \wedge G)$.

Similarly to how topological spectra define (co)homology theories on topological spaces, motivic spectra define (co)homology theories on Sm/k, and more generally on $\mathcal{SH}(k)$.

Definition 1.1.10 ([Wei94, Definition 10.2.7], [Nee01, Definition 1.1.7])**.** Let $\mathcal{A}$ be an abelian category. A homological functor H ([Wei94] calls this a covariant cohomological functor) is a functor $H : \mathcal{SH}(k) \to \mathcal{A}$ such that for distinguished triangles

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

in $\mathcal{SH}(k)$, the induced long sequence

$$\cdots \xrightarrow{h_*} H(\Sigma^i A) \xrightarrow{f_*} H(\Sigma^i B) \xrightarrow{g_*} H(\Sigma^i C) \xrightarrow{h_*} H(\Sigma^{i+1} A) \xrightarrow{f_*} \cdots$$

is exact. A cohomological functor is defined similarly.

Definition 1.1.11. Given k-spectra E and X, the $(m + n\alpha)$-th E-cohomology (respectively E-homology) of X is

$$E^{m+n\alpha}X := [X, S^{m+n\alpha} \wedge E] = [X, E]_{-(m+n\alpha)} \quad \text{(respectively} \quad E_{m+n\alpha}X := [S^{m+n\alpha}, E \wedge X]).$$

We use $E^\star X$ (respectively $E_\star X$) to denote the bigraded object of the E-cohomology groups (respectively E-homology groups) of X. For the sphere spectrum we will be even briefer, and define $E^\star := E^\star S$ and $E_\star := E_\star S$. It will always be clear from the context whether we refer to $E^\star$ as a functor or as the E-cohomology groups of S. Notice in particular that $E_\star = \pi_\star E = S_\star E$.

By [Wei94, Example 10.2.8], E-homology and E-cohomology define homological and cohomological functors. Hence, for cofiber sequences there are induced long exact sequences of E-(co)homology groups. We record this as a theorem for future reference.

Theorem 1.1.12. *Let E be a spectrum, and $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ a cofiber sequence in* $\mathrm{Spt}(k)$*. Then there are long exact sequences in E-homology*

$$\cdots \to E_{m+n\alpha}X \to E_{m+n\alpha}Y \to E_{m+n\alpha}Z \to E_{m+n\alpha-1}X \to \cdots$$

and in E-cohomology

$$\cdots \leftarrow E^{m+n\alpha}X \leftarrow E^{m+n\alpha}Y \leftarrow E^{m+n\alpha}Z \leftarrow E^{m+n\alpha-1}X \leftarrow \cdots$$

The category $\mathcal{SH}(k)$ is a closed monoidal category, hence, has an internal hom functor. We denote this functor by $\mathbf{Hom}(-,-)$. Some of the properties of $\mathbf{Hom}$ important to us are given in the following lemma.

Lemma 1.1.13. *Let $\mathbf{Hom}$ denote the internal hom functor, then:*

1. *$\mathbf{Hom}(X,-)$ is right adjoint to the smash product.*
2. *$\pi_\star(\mathbf{Hom}(X,Y)) = [X,Y]_\star$.*
3. *$\mathbf{Hom}(S,X) \cong X$.*
4. *$\mathbf{Hom}(X,-)$ commutes with suspension of both spheres.*
5. *$\mathbf{Hom}(\Sigma^{m+n\alpha}X,Y) = \Sigma^{-m-n\alpha}\mathbf{Hom}(X,Y)$.*

Proof. Property (1) is true by the definition of a closed monoidal category. The other properties are immediate from (1) and Lemma 1.1.8. □

In topology compact spaces are certainly very useful objects. Similarly we have a notion of compact motivic spectra.

Definition 1.1.14. A motivic spectrum X is compact if for any filtered colimit of motivic spectra, the canonical map $\mathrm{colim}[X,Y_i] \to [X, \mathrm{colim}\, Y_i]$ is an isomorphism.

Some examples of compact objects are given in the following lemma.

Lemma 1.1.15 ([Jar00, Lemma 2.2])**.** *An object that is compact in* $\mathrm{Spc}_\bullet(k)$ *remains compact in* $\mathrm{Spt}(k)$*.*

1. *If $A \hookrightarrow B$ is an inclusion of schemes, then the quotient A/B is compact.*
2. *All finite pointed simplical sets are compact.*
3. *All pointed schemes are compact.*
4. *If X_1 and X_2 are compact, then $X_1 \vee X_2$ and $X_1 \wedge X_2$ are compact.*
5. *If $g : X_1 \to X_2$ is a map of compact objects, then the cofiber is compact.*

In particular the motivic spheres are compact.

Lemma 1.1.16. *For a collection $\{E_i\}_i$ of spectra and F a spectrum we have*

$$\prod_i [E_i, F] = \left[\bigvee_i E_i, F\right].$$

If E is a compact object and F_i a filtered system of spectra then

$$\mathrm{colim}_i[E,F_i] = [E, \mathrm{hocolim}_i\, F_i].$$

In particular, $\bigoplus_i [E,F_i] \cong [E, \bigvee_i F_i]$ and $\mathrm{colim}\, \pi_\star A_i = \pi_\star\, \mathrm{hocolim}_i\, A_i$.

Proof. The first claim is a standard consequence of the duality of the product and coproduct. The second statement is true by definition of compact objects. □

Since the homotopy groups $\underline{\pi}_\star E$ are presheaves of abelian groups we might suspect that the stable hom groups should be presheaves. We get this if we define

$$\underline{[E,F]}_{m+n\alpha} : U \mapsto \underline{[E \wedge U_+, F]}_{m+n\alpha}.$$

Analogous results to Lemma 1.1.16, Theorem 1.1.12 and Lemma 1.1.13 holds for these presheaves. In this notation, $\underline{[E,F]}_{m+n\alpha}(\operatorname{Spec} k) = [E,F]_{m+n\alpha}$. This agrees with the previous definition of $\underline{\pi}_\star$ by a generalized variant of [Voe98, Theorem 5.2].

A useful aspect of spectra is the Brown representability theorem.

Proposition 1.1.17 (Brown representability, [Voe98, Proposition 5.4])**.** *Let k be a countable field. Given a cohomological functor $H : \mathcal{SH}(k) \to \mathcal{A}$, there exists a spectrum E such that $E^\star = H$. That is, all cohomology theories are representable by spectra.*

By Brown repsentability, cohomology theories on Sm/k which extend to cohomology theories on $\mathcal{SH}(k)$, are represented by spectra. In Section 1.3 we consider some cohomology theories and their representation by spectra.

1.2 Motivic Cohomology

In this section we give a short overview of motivic cohomology. We mostly refer to [MVW06] for the proofs. Throughout this section X denotes a smooth scheme over k and A an abelian group (outside of this section A is usually $\mathbb{Z}/2$ unless stated otherwise). In this section we use the (p,q)-grading (cf. the discussion on grading below Equation (1.1)), since this is the most common grading convention when dealing with motivic cohomology.

Motivic cohomology with A-coefficients is a collection of contravariant functors from smooth schemes to abelian groups

$$H^{p,q}(-,A) : (\mathrm{Sm}/k)^{\mathrm{op}} \to \mathrm{Ab}, \quad p,q \in \mathbb{Z}.$$

In Section 1.3 we show that these functors are represented by a spectrum. Hence they satisify certain $\mathbb{A}^1$-homotopy properties, similar to the way singular cohomology satisfy homotopy properties in topology.

We include the construction of motivic cohomology to give a taste of the subject. It will not be used anywhere else, except in the proof of Lemma 1.2.3 and Proposition 1.2.6.

Definition 1.2.1 ([MVW06, Definition 3.4])**.** Motivic cohomology with $\mathbb{Z}$-coefficients is defined as the hypercohomology [Wei94, 5.7.10] of $\mathbb{Z}(q)$ with resepct to the Zariski topology,

$$H^{p,q}(X,\mathbb{Z}) := \mathbb{H}^p(X, \mathbb{Z}(q)).$$

Here $\mathbb{Z}(q)$ denotes a certain complex of presheaves of transfers (see [MVW06, Definition 2.1]). Actually it is a complex of sheaves in both the Zariski, Nisnevich and étale topology on Sm/k. Other coefficients are obtained by tensoring the complex with A and taking hypercohomology.

From [Wei94, Application 5.7.10] there is a spectral sequence

$$E_2^{s,t} = H^s(X, H^t(\mathbb{Z}(q))) \implies H^{s+t,q}(X,\mathbb{Z}),$$

where H^s is ordinary sheaf cohomology in the Zariski topology, and $H^t(\mathbb{Z}(q))$ is the sheaf obtained by computing the t-th homology group of the complex. The spectral sequence is strongly convergent. This is a consequence of [Har77, III, Theorem 2.7] which implies that the E_2-page is bounded, since each $X \in \mathrm{Sm}/k$ has finite dimension. Hence, $H^i(X,\mathcal{F}) = 0$ for $i > \dim X$ and any sheaf $\mathcal{F}$.

Remark 1.2.2. Hypercohomology is the same in both the Zariski and Nisnevich topology. Hence, we could just as well have used the Nisnevich topology in place of the Zariski topology in Definition 1.2.1.

In the proof of Lemma 1.2.3 we use some of the results in Chapter 3 on convergence of spectral sequences.

Lemma 1.2.3. *Motivic cohomology commutes with filtered colimits in the coefficients,*

$$\operatorname{colim}_i H^{p,q}(X, A_i) = H^{p,q}(X, \operatorname{colim} A_i).$$

Proof. The morphisms $A_i \to \operatorname{colim}_i A_i$ induces morphisms $H^{p,q}(X, A_i) \to H^{p,q}(X, \operatorname{colim}_i A_i)$ compatible with the filtration implicit in the strong convergence of the spectral sequence above. Hence, we get a morphism $\operatorname{colim}_i H^{p,q}(X, A_i) \to H^{p,q}(X, \operatorname{colim} A_i)$. The E_2-page is

$$\operatorname{colim}_i (E_2^{s,t})_i = \operatorname{colim}_i H^s(X, H^t(\mathbb{Z}(q) \otimes A_i)) \to H^s(X, H^t(\mathbb{Z}(q) \otimes \operatorname{colim}_i A_i)).$$

Since colim commutes with sheaf cohomology ([Har77, III, Proposition 2.9]) and tensor products, this map is an isomorphism. Hence, the spectral sequences are isomorphic from the E_2-page and onwards, and the abutments are isomorphic by Theorem 3.1.5. □

Motivic cohomology satisfy some vanishing properties and is related to some classical objects of study. This is summarized below.

Proposition 1.2.4 ([MVW06, Theorem 3.6, 19.3, Corollary 4.2])**.** *Let X be a smooth scheme. For the motivic cohomology of X we have:*

- $H^{p,q}(X, A) = 0$, $p > q + \dim X$. *Here* $\dim X$ *is the dimension of X in the Zariski topology. In particular, when $X = \operatorname{Spec} k$, $H^{p,q} = 0$ below the diagonal $p = q$. This is a consequence of the spectral sequence in Definition 1.2.1.*
- $H^{p,q}(X, A) = 0$, *for $p > 2q$,*
- *For X a connected scheme we have*

$$H^{p,0}(X, A) = \begin{cases} A & p = 0, \\ 0 & p \neq 0. \end{cases}$$

-

$$H^{p,q}(X, \mathbb{Z}) = \begin{cases} \mathcal{O}^*(X) & p = 1, \\ \operatorname{Pic}(X) & p = 2, \\ 0 & p \neq 1, 2. \end{cases}$$

Here $\mathcal{O}^*$ is the sheaf of invertible elements in the structure sheaf [Har77, p. 141], *and* $\operatorname{Pic}(X)$ *is the Picard group of X* [Har77, p. 143].

Essential to us is the motivic cohomology of a point.

Proposition 1.2.5 ([MVW06, Theorem 5.1])**.** *For a field k we have the natural isomorphism*

$$H^{p,p}(\operatorname{Spec} k, A) = K_p^M(k) \otimes A.$$

(Milnor K-theory of fields, $K_p^M(k)$, is defined in Definition 2.1.1).

When $A = \mathbb{Z}/2$, there is a canonical element $\tau \in H^{0,1}$. With the product structure on $H^\star$ defined below, multiplication by τ^i is an isomorphism [DI10, Remark 4.4]. *Combined with the vanishing properties above this gives the full structure of $H^\star$,*

$$H^\star(\operatorname{Spec} k, \mathbb{Z}/2) = k_*^M(k)[\tau],$$

where $k_m^M(k)$ is in bidegree (m, m).

Another canonical element of $H^\star(\operatorname{Spec} k, \mathbb{Z}/2)$ *is ρ in bidegree* $(1, 1)$ *representing the class of $-1 \in k_1^M(k) = k^\times/(k^\times)^2$. When -1 has a square root, ρ is trivial.*

There is a universal coefficient theorem for motivic cohomology.

Proposition 1.2.6 ([MVW06, Lecture 4])**.** *For a group A there is a natural short exact sequence*

$$0 \to H^{p,q}(X,\mathbb{Z}) \otimes A \to H^{p,q}(X,A) \to \operatorname{Tor}_1^{\mathbb{Z}}(H^{p+1,q}(X,\mathbb{Z}),A) \to 0.$$

Proof. Consider the short exact sequence of complexes of presheaves of transfers

$$0 \to \mathbb{Z}(q) \xrightarrow{l} \mathbb{Z}(q) \to \mathbb{Z}(q)/l \to 0.$$

This induces a long exact sequence in hypercohomology [Wei94, 5.7.5],

$$\begin{tikzcd}
\cdots \arrow[r, "l"] & \mathbb{H}^p(X,\mathbb{Z}(q)) \arrow[rr] \arrow[dr, two heads] & & \mathbb{H}^p(X,\mathbb{Z}(q)/l) \arrow[rr] \arrow[dr, two heads] & & \mathbb{H}^{p+1}(X,\mathbb{Z}(q)) \arrow[r, "l"] & \cdots \\
& & \mathbb{H}^p(X,\mathbb{Z}(q))/l \arrow[ur, tail] & & \mathbb{H}^{p+1}(X,\mathbb{Z}(q))_l \arrow[ur, tail] & &
\end{tikzcd}$$

Since all the functors involved are additive, the maps remain multiplication by l in the long exact sequence. The general statement follows since any abelian group is the colimit of finitely generated groups, colim is exact and commutes with tensor-products, $\operatorname{Tor}_1^{\mathbb{Z}}$ and the coefficients in motivic cohomology. $\square$

Example 1.2.7. In this example we calculate $H^{a,b}(\operatorname{Spec}\mathbb{F}_q;\mathbb{Z}_2)$. There is a spectral sequence [RW00, Equation 1.2]:

$$E_2^{a,b} = \begin{cases} H_{\text{ét}}^{a-b}(k;\mathbb{Z}/2^\nu(-q)) & b \le a \le 0, \\ 0 & \text{otherwise} \end{cases} \implies K_{-a-b}(k;\mathbb{Z}/2^\nu).$$

Here $H_{\text{ét}}^a(k;\mathbb{Z}/2^\infty(b))$ is the étale cohomology of the constant sheaf $\mathbb{Z}/2^\infty$. See [Wei13, IV.2] for the definition of K-theory with finite coefficients, $K(k;\mathbb{Z}/2^\nu)$. Étale cohomology is related to motivic cohomology [RW00, p. 7]:

$$H^{a,b}(k;\mathbb{Z}/2^\nu) \cong \begin{cases} H_{\text{ét}}^a(k;\mathbb{Z}/2^\nu(b)) & 0 \le a \le b, \\ 0 & \text{otherwise.} \end{cases}$$

Let $k = \mathbb{F}_q$. From [Qui72] we know the algebraic K-theory of finite fields for $i \ge 0$ to be,

$$K_i(\operatorname{Spec} k) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z}/(q^k - 1) & i = 2k-1, \\ 0 & \text{otherwise.} \end{cases}$$

The universal coefficient theorem applied to K-theory with coefficients ([Wei13, IV.2.5]) implies that for $i \ge 0$ we have

$$K_i(\operatorname{Spec} k;\mathbb{Z}/2^\nu) = \begin{cases} \mathbb{Z}/2^\nu & i = 0, \\ \mathbb{Z}/2^\nu \otimes \mathbb{Z}/(q^k-1) & i = 2k-1, \\ \mathbb{Z}/2^\nu \otimes \mathbb{Z}/(q^k-1) & i = 2k,\ k > 0. \end{cases}$$

The cohomological dimension of finite fields are 1 [Mil08, Chapter 15]. This implies that the E_2-page of the spectral sequence is zero, except along $p - q = 0$ and $p - q = 1$. Hence, there are no differentials, and since we know the abutment we can read off the E_2-page:

$$H^{0,b}(k;\mathbb{Z}/2^\nu) = K_{2b}(k;\mathbb{Z}/2^\nu),$$
$$H^{1,b}(k;\mathbb{Z}/2^\nu) = K_{2b-1}(k;\mathbb{Z}/2^\nu),\ b \ge 1.$$

In all other degrees $H^{a,b}(k;\mathbb{Z}/2^\nu)$ is zero. We then obtain $H^{0,b}(k;\mathbb{Z}_2)$ from a variant of the universal coefficient theorem (Proposition 1.2.6):

$$0 \to \mathbb{Z}/2^\nu \otimes H^{a,b}(k;\mathbb{Z}_2) \to H^{a,b}(k;\mathbb{Z}/2^\nu) \to \mathrm{Tor}_1^{\mathbb{Z}}(H^{a+1,b}(k;\mathbb{Z}_2), \mathbb{Z}/2^\nu) \to 0.$$

Hence, we know $\mathbb{Z}/2^\nu \otimes H^{a,b}(k;\mathbb{Z}_2)$ for all $\nu \geq 1$. A lim-lim^1 argument then leaves only one possibility,

$$H^{a,b}(k,\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & (a,b) = (0,0), \\ \mathbb{Z}_2 \otimes \mathbb{Z}/(q^b - 1) & (a,b) = (1,b),\ b \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

When A is a ring, motivic cohomology can be given a homotopy associative pairing (i.e., $\mathbb{A}^1$-homotopy) [MVW06, Corollary 3.12]

$$H^{p,q}(X,A) \otimes H^{p',q'}(X,A) \to H^{p+p,q+q'}(X,A). \tag{1.2}$$

This pairing on $H^{*,*}$ is graded commutative with respect to the first grading [MVW06, 15.9], but this is mostly irrelevant to us, since we usually have $A = \mathbb{Z}/2$. For $X = \operatorname{Spec} k$, the pairing agree with the product structure from Proposition 1.2.5 on $K_p^M(k)$, see [Wei99]. We will elaborate on this in Section 1.3, when we consider the Eilenberg-Maclane spectrum representing motivic cohomology.

1.3 Some Motivic Spectra

In addition to the sphere spectrum, three other spectra are of particular interest to us.

- $H\mathbb{Z}$ and $H\mathbb{Z}/m$ the Eilenberg-Maclane spectra, representing motivic cohomology with $\mathbb{Z}$ and $\mathbb{Z}/m$ coefficients.
- **KGL**, representing algebraic K-theory of schemes.
- **KO**, representing Hermitian K-theory of schemes.

As detailed in Section 1.2 and Chapter 2, motivic cohomology, algebraic K-theory and Hermitian K-theory are cohomology functors on Sm/k. If they extend to cohomology functors on $\mathcal{SH}(k)$, Brown representability would tell us that they are represented by spectra in $\mathcal{SH}(k)$. However, the standard way to check that these theories can be extended to $\mathcal{SH}(k)$ is by constructing actual spectra representing them. We follow [Voe98] and [Hor05] and give these explicit descriptions.

Motivic cohomology

The motivic Eilenberg-Maclane spectrum represents motivic cohomology. Below we sketch the construction in [Voe98] with some extra details from [Dun+07].

For every scheme $X \in \mathrm{Sm}/k$ define a functor $L(X) : (\mathrm{Sm}/k)^{\mathrm{op}} \to \mathrm{Ab}$, by mapping a scheme U to the free abelian group on the finite correspondences of $U \times X$. Finite correspondences are the closed irreducible subsets $Z \subset U \times X$ which are surjective and finite over U. A map $f : U \to V$ is mapped to a map $L(X)f : Z \mapsto f^{-1}(Z)$. The functor $L(X)$ corresponds to the hom sets in the category of finite correspondences, Cor_k, as defined in [MVW06, Lecture 1]. They also prove that $L(X)$ is a Nisnevich sheaf.

There is a map of spaces $\Gamma(X) : X \to L(X)$, given by

$$X(U) \ni (f : U \to X) \mapsto \Gamma_f \in L(X)(U).$$

Here $\Gamma_f := \mathrm{im}(U \times f : U \to U \times X)$ is the graph of f. We proceed to extend the $L(X)$-construction to a functor $\mathrm{Spc}_\bullet \to \mathrm{Spc}_\bullet$ through a series of steps. The map $\Gamma(X) : X \to L(X)$ comes along for the journey.

The $L(X)$-construction extends to a functor $L : \mathrm{Sm}/k \to \mathrm{Pre}(\mathrm{Sm}/k, \mathrm{Ab})$. A morphism $f : X \to Y$ is mapped to $\mathrm{id} \times f : L(X)(U) \to L(Y)(U)$. This extends further to a functor $\mathrm{Pre}_\bullet(\mathrm{Sm}/k) \to \mathrm{Pre}(\mathrm{Sm}/k, \mathrm{Ab})$, since any element of $\mathrm{Pre}_\bullet(\mathrm{Sm}/k)$ is the colimit of representable functors. That is, if $\mathcal{F} = \mathrm{colim}_i \mathrm{Hom}(-, X_i)$, we set $L(\mathcal{F}) := \mathrm{colim}\, L(X_i)$. This is well defined since L commutes with colimits as a functor from Sm/k.

Finally we extend L to a functor $\Delta^{\mathrm{op}}\mathrm{Pre}_\bullet(\mathrm{Sm}/k) \to \Delta^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}/k, \mathrm{Ab})$. The forgetful functor $\mathrm{Ab} \to \mathrm{Set}_\bullet$, induces a functor $F : \Delta^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}/k, \mathrm{Ab}) \to \Delta^{\mathrm{op}}\mathrm{Pre}_\bullet(\mathrm{Sm}/k) = \mathrm{Spc}_\bullet(k)$. Hence, we get a functor $L := FL : \mathrm{Spc}_\bullet(k) \to \mathrm{Spc}_\bullet(k)$.

For every $m \in \mathbb{Z}$, define a functor $\mathrm{Ab} \to \mathrm{Ab}, G \mapsto G/mG$. This extends to a functor $F_m : \Delta^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}/k, \mathrm{Ab}) \to \Delta^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}/k, \mathrm{Ab})$. Set $L_m := FF_mL$. With these functors we define the motivic Eilenberg-Maclane spaces with coefficients $\mathbb{Z}$ and $\mathbb{Z}/m$.

Definition 1.3.1. The n-th motivic Eilenberg-Maclane space with $\mathbb{Z}$-coefficients (respectively, $\mathbb{Z}/m$-coefficients) are

$$K(\mathbb{Z}(n), 2n) := L((\mathbb{P}^1, \infty)^{\wedge n}) \ \ (\text{respectively,} \ \ K(\mathbb{Z}/m(n), 2n) := L_m((\mathbb{P}^1, \infty)^{\wedge n})).$$

For schemes X and Y there is a bilinear morphism $L(X) \times L(Y) \to L(X \times Y)$, given by external product of cycles. This extends to $\mathrm{Spc}_\bullet$, and gives pairings $L(X) \wedge L(Y) \to L(X \wedge Y)$. In particular, for the Eilenberg-Maclane spaces we obtain morphisms

$$m_{m,n} : K(\mathbb{Z}(n), 2n) \wedge K(\mathbb{Z}(m), 2m) \to K(\mathbb{Z}(n + m), 2n + 2m).$$

Definition 1.3.2. The Eilenberg-Maclane spectrum $H\mathbb{Z}$ is the spectrum with constituent spaces $H_n\mathbb{Z} := K(\mathbb{Z}(n), 2n)$ and structure maps

$$(\mathbb{P}^1, \infty) \wedge K(\mathbb{Z}(n), 2n) \xrightarrow{\Gamma(\mathbb{P}^1)\wedge \mathrm{id}} K(\mathbb{Z}(1), 2) \wedge K(\mathbb{Z}(n), 2n) \xrightarrow{m_{1,n}} K(\mathbb{Z}(n + 1), 2n + 2).$$

The Eilenberg-Maclane spectrum $H\mathbb{Z}/m$ has constituent spaces $H_n\mathbb{Z}/m := K(\mathbb{Z}/m(n), 2n)$ and structure maps

$$(\mathbb{P}^1, \infty) \wedge K(\mathbb{Z}/m(n), 2n) \xrightarrow{\Gamma(\mathbb{P}^1)\wedge \mathrm{id}} K(\mathbb{Z}/m(1), 2) \wedge K(\mathbb{Z}/m(n), 2n) \xrightarrow{m_{1,n}} K(\mathbb{Z}(n + 1), 2n + 2).$$

The adjoints of the structure maps are $\mathbb{A}^1$-weak equivalences ([Voe98, Theorem 6.2]). Hence, $H\mathbb{Z}$ and $H\mathbb{Z}/m$ are $\Omega_{\mathbb{P}^1}$-spectra.

From Definition 1.1.11, $H\mathbb{Z}$ and $H\mathbb{Z}/m$ have associated homology theories. We call these homology theories motivic homology.

The pairings $m_{m,n}$ extend to a pairing $H\mathbb{Z} \wedge H\mathbb{Z} \to H\mathbb{Z}$ [Voe03], making $H\mathbb{Z}$ a commutative ring spectrum. The same is true for $H\mathbb{Z}/m$. This pairing is the same as the one in Equation (1.2). This provides $H\mathbb{Z}^\star$ with the structure of a commutative $\mathbb{Z}$-algebra (respectively, $\mathbb{Z}/m$-algebra) [Voe03, Theorem 2.2]. For every spectrum F, the pairing gives $H\mathbb{Z}^\star F$ the structure of a $H\mathbb{Z}^\star$-module [Voe03, Corollary 2.3].

As in topology, a ring spectrum makes its associated homology theory into modules and comodules over the stable homotopy groups and over the homology of the spectrum itself [Koc96, Section 4.5]. More precisely, for an Eilenberg-Maclane spectrum H and a spectrum X we get a left $H_\star$-module structure on $H_\star X$ from (recall that we have a map $\pi_\star X \otimes \pi_\star Y \to \pi_\star(X \wedge Y)$ given by the smash product of maps)

$$H \wedge H \wedge X \xrightarrow{\mu\wedge X} H \wedge X.$$

Similarly we obtain a right $H_\star$-module structure on $H_\star H$ from the map

$$H \wedge H \wedge H \xrightarrow{H\wedge \mu} H \wedge H.$$

This structure makes it possible to consider the tensor product $H_\star H \otimes_{H_\star} H_\star X$.

Lemma 1.3.3 ([HKØ13, Proposition 5.5]). *The natural map*

$$H_\star H \otimes_{H_\star} H_\star E \to H_\star(H \wedge E)$$

induced by the smash product is an isomorphism.

With this theorem we can provide $H_\star X$ with a left $H_\star H$-module structure and a left $H_\star H$-comodule structure. The left module structure is induced by

$$H \wedge H \wedge H \wedge X \xrightarrow{H \wedge T \wedge X} H \wedge H \wedge H \wedge X \xrightarrow{\mu \wedge H \wedge X} H \wedge H \wedge X \xrightarrow{\mu \wedge X} H \wedge X.$$

The left comodule structure is obtained from

$$S \wedge X \xrightarrow{\eta \wedge X} H \wedge X.$$

We can also give $H^\star X$ a left $H^\star H$-module structure by composition of maps (cf. [Koc96, Proposition 4.5.4]), i.e.,

$$[H,H]_{-\star} \otimes [X,H]_{-\star} \to [X,H]_{-\star}, \;\; f \otimes g \mapsto fg.$$

From now on we make the convention that H denotes the motivic Eilenberg-Maclane spectrum $H\mathbb{Z}/2$. That is

$$H := H\mathbb{Z}/2.$$

Algebraic K-theory

Algebraic K-theory of schemes is defined in Definition 2.3.1. Below we sketch the construction in [Voe98, Section 6.2] of a spectrum **KGL** which represents algebraic K-theory.

Let $\mathrm{Gr}_n(\mathbb{A}^{n+m})$ be a Grassmannian. We then have canonical inclusions

$$\begin{aligned} &\mathrm{Gr}_n(\mathbb{A}^{n+m}) \hookrightarrow \mathrm{Gr}_n(\mathbb{A}^{n+m+1}), \\ &\mathrm{Gr}_n(\mathbb{A}^{n+m}) \hookrightarrow \mathrm{Gr}_{n+1}(\mathbb{A}^{n+m+1}), \quad L \mapsto L \oplus \{0\}. \end{aligned}$$

Consider the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\
\cdots \arrow[r] & \mathrm{Gr}_n(\mathbb{A}^{n+m}) \arrow[r] \arrow[d] & \mathrm{Gr}_n(\mathbb{A}^{n+m+1}) \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathrm{BGL}_n \arrow[d] \\
\cdots \arrow[r] & \mathrm{Gr}_{n+1}(\mathbb{A}^{n+m+1}) \arrow[r] \arrow[d] & \mathrm{Gr}_{n+1}(\mathbb{A}^{n+m+2}) \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathrm{BGL}_{n+1} \arrow[d] \\
 & \vdots & \vdots & & \vdots \arrow[d] \\
 & & & & \mathrm{BGL}
\end{tikzcd}$$

Here $\mathrm{BGL}_n := \mathrm{colim}_m \mathrm{Gr}_n(\mathbb{A}^{n+m})$, and $\mathrm{BGL} := \mathrm{colim}_n \mathrm{BGL}_n$. Define $\mathbb{Z} \times \mathrm{BGL} := \coprod_{i \in \mathbb{Z}} \mathrm{BGL}$ and let KGL be a fibrant replacement. The space KGL are the constituent spaces of **KGL**.

The structure maps $\mathbb{P}^1 \wedge \mathrm{KGL} \to \mathrm{KGL}$ are obtained from the following isomorphism ([Voe98, p. 600])

$$\mathrm{Hom}_{\mathcal{H}(k)}(\mathbb{P}^1 \wedge (\mathbb{Z} \times \mathrm{BGL}), \mathbb{Z} \times \mathrm{BGL}) \cong \mathrm{Hom}_{\mathcal{H}(k)}(\mathbb{Z} \times \mathrm{BGL}, \mathbb{Z} \times \mathrm{BGL}).$$

Under this identification the indentity morphism of $\mathbb{Z} \times \mathrm{BGL}$ provides us with the required map $\mathbb{P}^1 \wedge \mathbb{Z} \times \mathrm{BGL} \to \mathbb{Z} \times \mathrm{BGL}$. Since KGL is fibrant it is possible to lift this map to a map in $\mathcal{SH}(k)$, $e : \mathbb{P}^1 \wedge \mathbb{Z} \times \mathrm{KGL} \to \mathbb{Z} \times \mathrm{KGL}$. The adjoint $\Omega_{\mathbb{P}^1} e$ is a weak equivalence.

Definition 1.3.4. The spectrum representing algebraic K-theory is the $\Omega_{\mathbb{P}^1}$-spectrum $\mathbf{KGL}$ with constituent spaces $\mathbf{KGL}_i = \mathrm{KGL}$ and structure maps

$$e : \mathbb{P}^1 \wedge \mathrm{KGL} \to \mathrm{KGL}.$$

The spectrum $\mathbf{KGL}$ is $(1+\alpha)$-periodic and we have a Bott periodicity map

$$\beta : \Sigma^{1+\alpha}\mathbf{KGL} \xrightarrow{\simeq} \mathbf{KGL}.$$

Note that $\mathbf{KGL}$ is a ring spectrum [PPR09, Theorem 2.2.1], i.e., we have maps $\eta : S \to \mathbf{KGL}$ and $\mu : \mathbf{KGL}\wedge\mathbf{KGL} \to \mathbf{KGL}$ subject to the usual associativity and left and right unit diagrams. This ring structure is compatible with the ring structure on algebraic K-theory.

$\mathbf{KGL}$ represents algebraic K-theory in the sense that for a scheme $X \in \mathrm{Sm}/k$ we have ([Voe98, Theorem 6.9], there is a sign error in the article)

$$K_{m-n}(X) = \mathrm{Hom}_{\mathcal{SH}(k)}(S^{m+n\alpha} \wedge X_+, \mathbf{KGL}).$$

In particular for $X = \operatorname{Spec} k$ we get $K_{m-n}(k) = \pi_{m+n\alpha}(\mathbf{KGL})$.

Motivic cobordism

We define the spectrum representing motivic cobordism, $\mathbf{MGL}$. We sketch the construction in [Dun+07, Section 3.3]. Recall the construciton of the Thom space of a vector bundle $\mathcal{E} \to X$ (Definition 1.1.1). Consider the tautological vector bundle of the Grassmannian $\gamma_{n,m} \to \mathrm{Gr}_n(\mathbb{A}^m)$. Taking colimits over m we get the universal n-dimensional vector bundle $\gamma_n \to \mathrm{BGL}_n$. The canonical map $\mathrm{BGL}_n \to \mathrm{BGL}_{n+1}$ induces a pull-back square

$$\begin{tikzcd}
\mathbb{A}^1 \times \gamma_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & \gamma_{n+1} \arrow[d] \\
\mathrm{BGL}_n \arrow[r] & \mathrm{BGL}_{n+1}
\end{tikzcd}$$

Here the bundle $\mathbb{A}^1 \times \gamma_n \to \mathrm{BGL}_n$ is obtained by taking the product with the trivial vector bundle $\mathbb{A}^1 \to \operatorname{Spec} k$. We get an induced map on Thom-spaces

$$\mathrm{Th}(\mathbb{A}^1 \times \gamma_n) \to \mathrm{Th}(\gamma_{n+1}).$$

By the properties of Thom-spaces (Lemma 1.1.2) we get a series of equalities

$$\mathrm{Th}(\mathbb{A}^1 \times \gamma_n) = \mathrm{Th}(\mathbb{A}^1) \wedge \mathrm{Th}(\gamma_n) = \mathbb{A}^1/(\mathbb{A}^1 - 0) \wedge \mathrm{Th}(\gamma_n) = \mathbb{P}^1 \wedge \mathrm{Th}(\gamma_n),$$

where the last equality is due to Proposition 1.1.3. Hence, we have obtained a structure map $e_n : \mathbb{P}^1 \wedge \mathrm{Th}(\gamma_n) \to \mathrm{Th}(\gamma_n)$.

Definition 1.3.5. The motivic cobordism spectrum is defined to be the $\Omega_{\mathbb{P}^1}$-spectrum $\mathbf{MGL}$ with constituent spaces $\mathbf{MGL}_n = \mathrm{Th}(\gamma_n)$ and structure maps

$$e_n : \mathbb{P}^1 \wedge \mathrm{Th}(\gamma_n) \to \mathrm{Th}(\gamma_{n+1}).$$

Motivic cobordism is a connective ring spectrum [Hoy13, Corollary 3.9].

Hermitian K-theory

The representability of Hermitian K-theory in the motivic stable homotopy category was first proved by Hornbostel [Hor05] who constructed a $4(1+\alpha)$-periodic $\Omega_{\mathbb{P}^1}$-spectrum $\mathbf{KO}$. We outline his construction below.

Remark 1.3.6. Another common notation for $\mathbf{KO}$ is $\mathbf{KQ}$.

The first step is to consider the functor K^h : Rings $\to$ Ab as defined in Definition 2.3.1 and extend it to a motivic space, $KO : (\mathrm{Sm}/k)^{\mathrm{op}} \to (\mathrm{Sm}/k)^{\mathrm{op}}$. This is done in Section 2 of [Hor05], and is sufficient to prove that Hermitian K-theory is representable in the motivic unstable homotopy category ([Hor05, Corollary 3.4]):

$$KO_n(X) \cong \mathrm{Hom}_{\mathcal{H}(k)}(S^n \wedge X_+, aKO_f).$$

Here aKO is the sheafification of KO in the Nisnevich topology, and aKO_f is a fibrant replacement of aKO. This proposition is used to extend aKO_f to a functor $\Delta^{\mathrm{op}}\mathrm{Shv}(\mathrm{Sm}/k)_{\mathrm{Nis},\bullet}$.

To prove that KO is representable in the stable homotopy category we need some more constructions. In Section 2.3 we construct topological spaces $K(X)$ and $K^h(X)$ of an affine scheme X, such that the homotopy groups represent ordinary and Hermitian K-theory of X. The hyperbolic functor (Section 2.2) induces a map of topological spaces $H : K(X) \to K^h(X)$. The homotopy fiber $U(X) := \mathrm{hofib}(H : K(X) \to K^h(X))$ defines a theory on affine schemes via $\pi_n(U(X))$. Similarly we consider the homotopy fiber $USp(X) := \mathrm{hofib}(\bar{H} : K(X) \to KSp(X))$. In the same fashion as for K^h, U is extended to spaces. In the end Hornbostel defines the spectrum $\mathbf{KO}$ such that

$$\mathbf{KO}_i = \begin{cases} aKO_f & i \equiv 0 \ (4), \\ aUSp_f & i \equiv 1 \ (4), \\ aKSp_f & i \equiv 2 \ (4), \\ aU_f & i \equiv 3 \ (4). \end{cases}$$

The structure maps are obtained by a series of isomorphisms resulting in an isomorphism similar to the one for $\mathbf{KGL}$,

$$\mathrm{Hom}_{\mathcal{H}(k)}(aU_f, aU_f) \cong \mathrm{Hom}_{\mathcal{H}(k)}(aU_f, \Omega_{\mathbb{P}^1} aKO_f),$$

which provides us with weak equivalences $e : \mathbf{KO}_{4i+3} = aU_f \to \Omega_{\mathbb{P}^1} aKO_f = \mathbf{KO}_{4i}$, and similarly for the other parts of the spectrum. The resulting spectrum is an $\Omega_{\mathbb{P}^1}$-spectrum and a ring spectrum [PW10, Theorem 1.5]. It represents Hermitian K-theory in the sense that for a scheme $X \in \mathrm{Sm}/k$ we have ([Hor05, Theorem 5.5])

$$KO_m(X) \cong \mathrm{Hom}_{\mathcal{SH}(k)}(S^m \wedge X_+, \mathbf{KO}).$$

More genereally we have

$$\mathrm{Hom}_{\mathcal{SH}(k)}(S^{m+n\alpha} \wedge X_+, \mathbf{KO}) \cong \begin{cases} KO_{m-n}(X) & n \equiv 0 \ (4), \\ USp_{m-n}(X) & n \equiv 1 \ (4), \\ KSp_{m-n}(X) & n \equiv 2 \ (4), \\ U_{m-n}(X) & n \equiv 3 \ (4). \end{cases}$$

In particular for $X = \mathrm{Spec}\, k$,

$$\pi_{m+n\alpha}(\mathbf{KO}) = \begin{cases} KO_{m-n}(k) & n \equiv 0 \ (4), \\ USp_{m-n}(k) & n \equiv 1 \ (4), \\ KSp_{m-n}(k) & n \equiv 2 \ (4), \\ U_{m-n}(k) & n \equiv 3 \ (4). \end{cases} \tag{1.3}$$

Algebraic K-theory can be identified as a cofiber of $\mathbf{KO}$ ([RØ13, Theorem 4.4]):

$$\Sigma^\alpha \mathbf{KO} \xrightarrow{\eta} \mathbf{KO} \xrightarrow{p} \mathbf{KGL} \xrightarrow{\delta} \Sigma^{1+\alpha}\mathbf{KO}. \tag{1.4}$$

Here η is the map induced by the Hopf map $\eta \in \pi_\alpha S$. Hence, we have a map of cofiber sequences

$$\begin{tikzcd} \Sigma^\alpha S \arrow[r, "\eta"] \arrow[d] & S \arrow[r, "p'"] \arrow[d] & C\eta \arrow[r, "\delta'"] \arrow[d] & \Sigma^{1+\alpha} S \arrow[d] \\ \Sigma^\alpha ko \arrow[r, "\eta"] & ko \arrow[r, "p"] & kgl \arrow[r, "\delta"] & \Sigma^{1+\alpha} ko \end{tikzcd} \tag{1.5}$$

The vertical maps are induced from the unit map $S \to ko$, and $C\eta$ is the cone of η.

The Brown-Peterson spectrum

For the computation of the cohomology of the spectra above we need a motivic version of the Brown-Peterson spectrum, **BP**, and its relation to **KGL**. In Definition 1.3.5 we defined the spectrum representing motivic cobordism, **MGL**. The Brown-Peterson spectrum is defined as a quotient of a localization (Definition 1.4.8) of this spectrum. We will not elaborate on the construction of quotients of **MGL** with respect to elements $(x_i)_{i\in\mathbb{N}} \subset \pi_\star\mathbf{MGL}$. Good accounts can be found in [Spi10, Section 4] and [Hoy13, Section 6.2]. Roughly the quotients are constructed iteratively by forming cofibers

$$\Sigma^{|x_n|}\mathbf{MGL}/(x_0,\dots,x_{n-1}) \xrightarrow{x_n} \mathbf{MGL}/(x_0,\dots,x_{n-1}) \to \mathbf{MGL}/(x_0,\dots,x_n),$$

and taking the hocolim of the result.

Definition 1.3.7. The motivic Brown-Peterson spectrum mod 2 is defined as

$$\mathbf{BP} := \mathbf{MGL}_{(2)}/(x_0,x_1,\dots),$$

for $(x_i)_{i\in\mathbb{N}}$ a maximal h-regular sequence in $\pi_\star\mathbf{MGL}$ [Hoy13]. By [Spi10, Remark 5.3] this definition agrees with the one of [Vez01].

From Section 1.3 we know that **MGL** is connective. Each quotient $\mathbf{MGL}/(x_n,\dots,x_{n+k})$ is connective, since it is the cofiber of a positive bidegree map between connective spectra. Hence, **BP** is a connective spectrum.

Lemma 1.3.8 ([Orm11])**.** *The motivic homotopy groups of* **BP** *contain canonical elements* $v_i \in \pi_{(2^i-1)(1+\alpha)}\mathbf{BP}$.

Taking further quotients of **BP** by these elements we arrive at the spectra $\mathbf{BP}\langle n\rangle$.

Definition 1.3.9. Let $\mathbf{BP}\langle n\rangle := \mathbf{BP}/(v_{n+1},v_{n+2},\dots)$.

There is a canonical map $\mathbf{BP} \to \mathbf{BP}\langle n\rangle$, which on homotopy groups maps $v_i \mapsto 0, i > n$.

Lemma 1.3.10. *The canonical map* $\mathbf{BP} \to \mathbf{BP}\langle n\rangle$ *induces a map* $\pi_{m+k\alpha}\mathbf{BP} \to \pi_{m+k\alpha}\mathbf{BP}\langle n\rangle$. *This map is an isomorphism for* $m < 2^{n+1}-1$ *and surjective for* $m = 2^{n+1}-1$. *In particular, since* **BP** *is connective, the spectra* $\mathbf{BP}\langle n\rangle$ *are connective.*

Proof. Consider the cofiber sequence

$$\Sigma^{|v_{n+1}|}\mathbf{BP} \xrightarrow{v_{n+1}} \mathbf{BP} \to \mathbf{BP}/(v_{n+1}).$$

Since **BP** is connective, the long exact sequence of homotopy groups implies that the map

$$\pi_{m+n\alpha}\mathbf{BP} \to \pi_{m+n\alpha}\mathbf{BP}/(v_{n+1})$$

is an isomorphism for $m < 2^{n+1}-1$, and surjective for $m = 2^{n+1}-1$. Continuing like this we get the same properties for the map

$$\pi_{m+n\alpha}\mathbf{BP} \to \pi_{m+n\alpha}\mathbf{BP}/(v_{n+1},v_{n+2},\dots,v_{n+k}).$$

Since $\mathbf{BP}\langle n\rangle = \operatorname{hocolim}_k \mathbf{BP}/(v_{n+1},v_{n+2},\dots,v_{n+k})$, colim is exact and commutes with $\pi_\star$, we obtain the lemma. □

The following theorem of Hoyois computes the motivic cohomology of $\mathbf{BP}\langle n\rangle$.

Theorem 1.3.11 ([Hoy13, Theorem 6.19])**.** *The motivic cohomology of the motivic Brown-Peterson spectrum and its quotients are*

$$H^\star\mathbf{BP} = A^\star/A^\star(Q_0,Q_1,\dots),$$
$$H^\star\mathbf{BP}\langle n\rangle = A^\star/A^\star(Q_0,Q_1,\dots,Q_n) = A^\star//E(n).$$

Here $A^\star$ *is the motivic Steenrod algebra defined in Section 1.7, the* Q_i*'s are Milnor primitives Definition 1.7.5,* $E(n)$ *and the quotient are defined in Definition 1.7.6.*

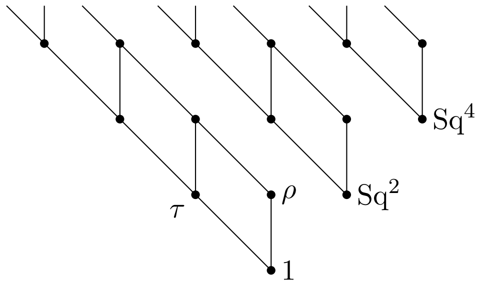


Figure 1.1: Cohomology of **BP** in low degrees. The diagonal lines are multiplication by τ, the vertical lines are multiplication by ρ.

Remark 1.3.12. Figure 1.1 is a picture of the cohomology of **BP** in low degrees over a base scheme such that $H^\star = \mathbb{F}_2[\tau,\rho]/(\rho^2)$ (e.g., our favourite base schemes $\operatorname{Spec}\mathbb{F}_q$, for q odd). Observe that in the first few dimensions $H^\star\mathbf{BP}$ is generated by 1, Sq^2 and Sq^4 over $H^\star$.

Lemma 1.3.13 ([Orm11, 3])**.** *There are cofiber sequences*

$$\Sigma^{|v_n|}\mathbf{BP}\langle n\rangle \xrightarrow{v_n} \mathbf{BP}\langle n\rangle \to \mathbf{BP}\langle n-1\rangle.$$

1.4 Completion and Localization

In this section we construct completion and localization of spectra with respect to prime numbers. First we discuss Moore spectra, which are used to define completion and localization. Our discussion is a bit more general than what we need. Outside of this section almost all completions and localizations are with respect to 2.

Moore spectra

Definition 1.4.1. Let A be an abelian group. A Moore spectrum of A is defined to be a spectrum SA such that $SA \wedge H\mathbb{Z} = HA$. Note that any Moore spectrum can be constructed as the colimit of spectra of the form S and S/n, since all abelian groups are the filtered colimit of finitely generated groups, and filtered colimits commute with smash-products and cohomology (Lemma 1.2.3).

By abuse of notation, we often write Moore spectra of the type $S\mathbb{Z}[\cdots]$ as $S[\cdots]$.

Let S be the sphere spectrum, and E any spectrum. Consider the multiplication by n map $S \xrightarrow{n} S$, $n = n{\cdot}\mathrm{id} \in [S,S]$. Note that this map induces multiplication by n in $[E,S] \xrightarrow{[E,n]=n} [E,S]$ and $[S,E] \xrightarrow{[n,E]=n} [S,E]$. The mod n Moore spectrum S/n is the cofiber of n,

$$S \xrightarrow{n} S \to S/n \to \Sigma S.$$

Lemma 1.4.2. *Let H be mod n motivic cohomology. Then $H^\star(S/n)$ is free on $H^\star(S)$ on two generators, x and y of bidegree* 0 *and* 1*, respectively.*

Proof. This follows from the long exact sequence in cohomology

$$\cdots \to H^\star\Sigma S \xrightarrow{\Sigma n^\star=0} H^\star\Sigma S \to H^\star S/n \to H^\star S \xrightarrow{n^\star=0} H^\star S \to \cdots,$$

which, because $n^\star = 0$, splits into short exact sequences

$$0 \to H^\star\Sigma S \to H^\star S/n \to H^\star S \to 0.$$

This is a sequence of $H^\star$-module maps, hence, it is split. The elements x and y are then elements defined by $H^0 S/n \ni x \mapsto 1 \in H^0 S$, and $H^1\Sigma S \ni 1 \mapsto y \in H^1 S/n$. $\square$

Example 1.4.3. From the commutative diagram

$$\begin{tikzcd} S \arrow[r, "n^k"] \arrow[d, "\mathrm{id}"] & S \arrow[r, "p"] \arrow[d, "n"] & S/n^k \arrow[d, dotted] \\ S \arrow[r, "n^{k+1}"] & S \arrow[r, "p"] & S/n^{k+1} \end{tikzcd}$$

we obtain maps $S/n^k \to S/n^{k+1}$. Putting all these together we get a diagram

$$S/n \to S/n^2 \to S/n^3 \to \cdots$$

The hocolim of this diagram is S/n^∞, the Moore spectrum of the divisible group $\mathbb{Z}/n^\infty := \mathbb{Z}\left[\frac{1}{n}\right]/\mathbb{Z}$.

Lemma 1.4.4. *Let E be any spectrum and let SA be a Moore spectrum of an abelian group A. Then we have two natural short exact sequences of presheaves of abelian groups*

$$\begin{gathered} 0 \to A \otimes \underline{\pi}_{p,q} E \to \underline{\pi}_{p,q}(SA \wedge E) \to \mathrm{Tor}^1_{\mathbb{Z}}(A, \underline{\pi}_{p-1,q} E) \to 0, \\ 0 \to \mathrm{Ext}^1_{\mathbb{Z}}(A, \underline{\pi}_{p+1,q} E) \to \underline{[SA, E]}_{p,q} \to \mathrm{Hom}_{\mathbb{Z}}(A, \underline{\pi}_{p,q} E) \to 0. \end{gathered}$$

Proof. Consider a free resolution of A,

$$0 \to C \xrightarrow{d} D \to A \to 0.$$

We get a cofiber sequence $SC \to SD \to SA$, where SC and SD are wedges of S, and the d' is the image of d through the bijection

$$\mathrm{Hom}_{\mathbb{Z}}(C, D) = \oplus_i \prod_j \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \to \oplus_i \prod_j [S, S]_0 = [SD, SC]_0.$$

If we apply $\underline{\pi}_\star(- \wedge E)$, we get a long exact sequence. The short exact sequence at $\underline{\pi}_{p,q}(SA \wedge E)$ is

$$\begin{aligned} 0 \to \mathrm{coker}\left(\underline{\pi}_{p,q} SC \wedge E \xrightarrow{\underline{\pi}_\star d' \wedge E} \underline{\pi}_{p,q} SD \wedge E\right) &\to \underline{\pi}_{p,q} SA \wedge E \\ \to \ker\left(\underline{\pi}_{p-1,q} SC \wedge E \xrightarrow{\underline{\pi}_\star d' \wedge E} \underline{\pi}_{p-1,q} SD \wedge E\right) &\to 0. \end{aligned}$$

From

$$\oplus_j\, \pi_{p,q} E = \underline{\pi}_{p,q} SC \wedge E \xrightarrow{d \otimes \pi_\star E} \underline{\pi}_{p,q} SD \wedge E = \oplus_i \underline{\pi}_{p,q} E$$

we get the first short exact sequence.

If we apply $[-, E]$ we get a long exact sequence. The short exact sequence at $[SA, E]_{p,q}$ is

$$\begin{aligned} 0 \to \mathrm{coker}\left([SD, E]_{p+1,q} \xrightarrow{[d', E]} [SC, E]_{p+1,q}\right) &\to [SA, E]_{p,q} \\ \to \ker\left([SD, E]_{p,q} \xrightarrow{[d', E]} [SC, E]_{p,q}\right) &\to 0. \end{aligned}$$

From

$$\prod_i \pi_{p,q} E = [SD, E]_{p,q} \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(d, \pi_\star E)} [SC, E]_{p,q} = \prod_j \pi_{p,q} E$$

we get the last short exact sequence. $\square$

Completion

With Moore spectra set up we are ready to define the n-completion of a spectrum E.

Definition 1.4.5. Let E be a spectrum. Then the n-completion of E is the spectrum

$$E_{\hat{n}} := \mathbf{Hom}(\Sigma^{-1}S/n^{\infty}, E).$$

The motivation behind this definition is that $E_{\hat{n}}$ is a S/n-fibrant replacement of E. That is, up to S/n-equivalence, E and $E_{\hat{n}}$ are equivalent, and $E_{\hat{n}}$ is S/n-fibrant. This again implies that the n-primary part of their homotopy groups are the same, if they are finitely generated. See [RØ08] for further details.

Lemma 1.4.6. *There is a short exact sequence of abelian presheaves*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n^{\infty}, \underline{\pi}_{p,q}(E)) \to \underline{\pi}_{p,q}(E_{\hat{n}}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n^{\infty}, \underline{\pi}_{p-1,q}(E)) \to 0.$$

Proof. From Lemma 1.1.13 we have $\underline{\pi}_{\star}(E_{\hat{n}}) = \underline{[\Sigma^{-1}S/n^{\infty}, E]}_{\star}$. Combined with Lemma 1.4.4 this yields the exact sequence. □

At the level of homotopy groups, when the groups $\pi_{p,q}(E)$ are finitely generated, the above lemma simplifies, and we have $\pi_{p,q}(E_{\hat{2}}) \cong \pi_{p,q}(E) \otimes \mathbb{Z}_2$, because of the following lemma:

Lemma 1.4.7 ([Wei94, Application 3.5.10])**.** *Let A be a abelian group which is finitely generated. Then*

1. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n^{\infty}, A) = 0$,
2. $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n^{\infty}, A) = \mathbb{Z}_n \otimes A$.

Proof. The group $\mathbb{Z}/n^{\infty}$ is divisible. This proves the first claim. From [Wei94, Application 3.5.10] we have an exact sequence

$$0 \to \lim{}^1_i \mathrm{Hom}(\mathbb{Z}/n^i, A) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n^{\infty}, A) \to \lim_i(\mathbb{Z}/n^i \otimes A) \to 0.$$

The Hom-groups are finite, since A is finitely generated, hence the inverse system satisfy the Mittag-Leffler condition. This imply that the $\lim^1$-term is zero. By a lim-$\lim^1$ argument we can move the tensor product out of the lim-term, and we obtain the second claim. □

Localization

Definition 1.4.8. Let S be a multiplicatively closed subset of $\mathbb{Z}\backslash\{0\}$. Define the $S^{-1}\mathbb{Z}$-localization of E to be the spectrum $E_{S^{-1}\mathbb{Z}} := S(S^{-1}\mathbb{Z}) \wedge E$. If $S = \mathbb{Z}\backslash(n)$ we set $E_{(n)} := E_{S^{-1}\mathbb{Z}}$.

The main properties of localization are contained in the following theorem.

Lemma 1.4.9 ([Bou79, Proposition 2.4])**.** *Let S be a multiplicatively closed subset of $\mathbb{Z}\backslash\{0\}$. Then localization with respect to S satisfy:*

1. *Localization commutes with* hocolim.
2. *Localization commutes with smash products.*
3. *Localization preserves cofiber sequences.*
4. *Localization commutes with formation of quotients (Section 1.3).*
5. *Localization commutes with f_q, the effective-functors (Definition 1.5.4), when $q \leq 0$.*
6. $\underline{\pi}_{\star}E_{S^{-1}\mathbb{Z}} = S^{-1}\mathbb{Z} \otimes \underline{\pi}_{\star}E$.
7. $(E_{S^{-1}\mathbb{Z}})_{S^{-1}\mathbb{Z}} \simeq E_{S^{-1}\mathbb{Z}}$.

8. $H_\star E_{S^{-1}\mathbb{Z}} = S^{-1}\mathbb{Z} \otimes H_\star E$.

Proof. Property (1), (2) and (3) follow immediately from the definition and Proposition 1.1.9. Property (4) is a consequence of (1) and (2) and the definition of quotients. Property (5) is contained in [Lev13a, Example B.5]. This can also be seen from the fact that $f_q S = S, q \leq 0$, and that f_q commutes with hocolim. Property (6) is a consequence of Lemma 1.4.4, since $S^{-1}\mathbb{Z}$ is torsion free. Finally, property (7) and (8) are immediate from Lemma 1.1.8 and (6). $\square$

Lemma 1.4.10. *Let E be a cellular spectrum (Definition 1.6.1) with finitely generated homotopy groups. Then the map $E_{\hat{\ell}} \to (E_{(\ell)})_{\hat{\ell}}$ is a stable equivalence.*

Proof. Consider the induced map on motivic homotopy groups

$$\pi_{m+n\alpha}\left(E_{\hat{\ell}}\right) \to \pi_{m+n\alpha}\left(\left(E_{(\ell)}\right)_{\hat{\ell}}\right).$$

By Lemma 1.4.4, Lemma 1.4.6 and Lemma 1.4.7 this simplifies to an isomorphism

$$\mathbb{Z}_\ell \otimes \pi_{m+n\alpha} E \to \mathbb{Z}_\ell \otimes (\pi_{m+n\alpha} E)_{(\ell)} \xrightarrow{\cong} \mathbb{Z}_\ell \otimes \pi_{m+n\alpha} E.$$

Hence, Proposition 1.6.3 implies the lemma. $\square$

1.5 Connective Spectra

In this section we discuss connective spectra, and how to construct the connected cover of a spectrum. Our model for the connected cover might not be the best and causes some problems, cf. Remark 4.1.1.

Definition 1.5.1 ([Hoy13, 2.]). Let $\mathcal{SH}(k)_{\geq d}$ be the smallest triangulated subcategory of $\mathcal{SH}(k)$ which is closed under direct sums and contains the spectra

$$\{\Sigma^{m+n\alpha}\Sigma^\infty X_+ | X \in \mathrm{Sm}/k, m \geq d, n \in \mathbb{Z}\}.$$

A spectrum in $\mathcal{SH}(k)_{\geq d}$ is called d-connective (or simply connective if $d = 0$). They are characterized by the following theorem of Hoyois.

Theorem 1.5.2 ([Hoy13, Theorem 2.3]). *Let E be a motivic spectrum. Then E is connective if and only if the presheaf of homotopy groups $\underline{\pi}_{m+n\alpha} E$ is zero for $m < 0$. That is*

$$E \in \mathcal{SH}(k)_{\geq 0} \iff \underline{\pi}_{m+n\alpha} E = 0, m < 0.$$

There is a variant of the Hurewicz-isomorphism in stable motivic homotopy theory.

Theorem 1.5.3. *If the map of spectra $E \to F$ induces an isomorphism $\pi_{m+n\alpha} E \to \pi_{m+n\alpha} F$ for $m + n < d$ and an epimorphism for $m + n = d$, then the map on motivic cohomology $H^{m+n\alpha} F \to H^{m+n\alpha} E$ is an isomorphism for $m + n < d$ and a monomorphism for $m + n = d$.*

We now introduce the slice filtration, (e.g., [Voe02b, 2] or [Hoy13, 8.3]). Let $\mathcal{SH}^{\mathrm{eff}}(k)$ be the smallest triangulated subcategory in $\mathcal{SH}(k)$ which is closed under direct sums and contains all suspension spectra of spaces, and their $\Sigma^{i(1+\alpha)}$-suspensions, $i \geq 0$ (i.e., $\mathbb{P}^1$-suspensions). This is the subcategory generated by homotopy colimits and extensions (i.e., formation of cofibers) by

$$\{\Sigma^{m+n\alpha}\Sigma^\infty X_+ | X \in \mathrm{Sm}/k, m \in \mathbb{Z}, n \geq 0\}.$$

The inclusions of the subcategories $\Sigma^{q(1+\alpha)}\mathcal{SH}^{\mathrm{eff}}(k), q \in \mathbb{Z}$, of $\mathcal{SH}(k)$ form a filtration of $\mathcal{SH}(k)$ called the slice filtration

$$\cdots \subset \Sigma^{(q+1)(1+\alpha)}\mathcal{SH}^{\mathrm{eff}}(k) \subset \Sigma^{q(1+\alpha)}\mathcal{SH}^{\mathrm{eff}}(k) \subset \Sigma^{(q-1)(1+\alpha)}\mathcal{SH}^{\mathrm{eff}}(k) \subset \cdots \subset \mathcal{SH}(k).$$

Definition 1.5.4. Let i_q be the canonical inclusion $i_q : \Sigma^{q(1+\alpha)}\mathcal{SH}(k) \hookrightarrow \mathcal{SH}(k)$. It has a right adjoint denoted by r_q ([Voe02b]). Together they form a functor $f_q = i_q r_q$. From adjointness there is a natural transformation, $f_q \to \mathrm{id}$, that is, for every spectrum E there is a universal map $f_q E \to E$, called the effective q-cover of E (by abuse of notation we also refer to $f_q E$ as the effective q-cover).

There are canonical maps $f_{q+1} = f_{q+1} f_q \to f_q$ obtained by composing $f_{q+1} \to \mathrm{id}$ with f_q. The equality $f_{q+1} = f_{q+1} f_q$ is a consequence of a theorem of Neemann ([Nee96, Theorem 4.1], the theorem used to construct the r_q's above) applied to the inclusion $\Sigma^{q+1}\mathcal{SH}^{\mathrm{eff}} \hookrightarrow \Sigma^q\mathcal{SH}^{\mathrm{eff}}$, and uniqueness of right adjoints. See for instance [Kel13, Section 4.2]. The functors f_q are triangulated and preserve homotopy colimits ([Hoy13, 8.3], [Spi10, Corollary 4.6]).

Lemma 1.5.5 ([RØ13, Lemma 2.1])**.** *There are natural isomorphisms*

$$f_{q+1}\Sigma^\alpha E \cong \Sigma^\alpha f_q E.$$

The isomorphisms are compatible with the natural transformations $f_{q+1} \to f_q$. That is, the following diagram is commutative

$$\begin{tikzcd} f_{q+1}\Sigma^\alpha E \arrow[r, "\cong"] \arrow[d] & \Sigma^\alpha f_q E \arrow[d] \\ f_q\Sigma^\alpha E \arrow[r, "\cong"] & \Sigma^\alpha f_{q-1} E \end{tikzcd}$$

Lemma 1.5.6 ([Kel13, Lemma 4.2.7])**.** *For any spectrum E we have*

$$E \simeq \mathrm{hocolim}_q(f_{-q+1}E \to f_{-q}E).$$

We define the connective cover of a spectrum E, to be its zeroth cover. We typically denote it by $e := f_0 E$. With this convention, let kgl and ko be the connective covers of $\mathbf{KGL}$ and $\mathbf{KO}$ localized at 2. That is

$$kgl := f_0(\mathbf{KGL}_{(2)}), \qquad ko := f_0(\mathbf{KO}_{(2)}).$$

Remark 1.5.7. Since the functor f_0 is triangulated, the cofiber sequence Equation (1.4) remains a cofiber sequence for the connected covers, that is, we have the following cofiber sequence on connective covers

$$\Sigma^\alpha ko \xrightarrow{\eta} ko \to kgl \to \Sigma^{1+\alpha} ko \tag{1.6}$$

(by abuse of notation we lef η denote $f_0\eta$).

The following theorem relates $\mathbf{BP}$ to kgl.

Proposition 1.5.8. *We have the following isomorphisms in $\mathcal{SH}(k)$:*

1. $\mathbf{BP}\langle 0\rangle \cong H\mathbb{Z}_{(2)}$,
2. $\mathbf{BP}\langle 1\rangle \cong kgl$.

In particular, by Theorem 1.3.11 we know the cohomology of $H\mathbb{Z}_{(2)}$ and $kgl_{(2)}$.

Proof. The first isomorphism is a corollary of [Hoy13, Theorem 7.12].

The second claim is a consequence of [Spi10, Proposition 5.4]. The assumptions in the proposition are fulfilled because of the first claim. The proposition provides us with a canonical map $\mathbf{BP}\langle 1\rangle \to kgl$. This map induces isomorphisms on the slice functors (Section 3.5). Hence, the slice spectral sequence is an isomorphism on the E_1-page, so by Proposition 3.5.1 and Theorem 3.1.5 the canonical map induces an isomorphism on sheaves of homotopy groups. Hence, the canonical map is a stable equivalence, and $\mathbf{BP}\langle 1\rangle = kgl$. □

We need the following two corollaries in Section 1.8.

Corollary 1.5.9. *The maps induced by* $\mathbf{BP} \to kgl$,

$$\pi_{m+n\alpha}\mathbf{BP} \to \pi_{m+n\alpha}kgl,$$
$$H^{m+n\alpha}kgl \to H^{m+n\alpha}\mathbf{BP}.$$

are isomorphisms for $m + n\alpha, m + n < 3$. *The first map is surjective for* $m + n = 3$ *and the second map is injective for* $m + n = 3$.

Proof. The first claim is a consequence of the identification with $\mathbf{BP}\langle 1\rangle$ in Proposition 1.5.8. The second claim is true by Theorem 1.5.3. $\square$

Corollary 1.5.10. *We have a cofiber sequence*

$$\Sigma^{1+\alpha}kgl \to kgl \to H\mathbb{Z}_{(2)}.$$

Lemma 1.5.11 ([Orm11, Theorem 3.3])**.** *We have an exact sequence*

$$0 \to \pi_\star kgl_{v_1} \to \pi_\star kgl \to \pi_\star \mathbf{KGL} \otimes \mathbb{Z}_{(2)},$$

where $\pi_\star kgl_{v_1}$ *denotes the* v_1 *torsion of* $\pi_\star kgl$. *For* $\star = m + n\alpha$, $m \geq 0$ *and* $m + n \geq 0$, *this is in fact a short exact sequence*

$$0 \to \pi_\star kgl_{v_1} \to \pi_\star kgl \to \pi_\star \mathbf{KGL} \otimes \mathbb{Z}_{(2)} \to 0.$$

Proof. This proof is a combination of [Orm11, Theorem 3.3] and remarks of [NSØ09, 3]. Bott-periodicity is

$$\Sigma^{1+\alpha}\mathbf{KGL} \xrightarrow[\beta]{\cong} \mathbf{KGL}.$$

If we apply f_{q+1} and Lemma 1.5.5 we get

$$f_{q+1}\mathbf{KGL} = f_{q+1}\Sigma^{1+\alpha}\mathbf{KGL} = \Sigma^{1+\alpha}f_q\mathbf{KGL},$$

and by induction $f_q\mathbf{KGL} = \Sigma^{q(1+\alpha)}f_0\mathbf{KGL} = \Sigma^{q(1+\alpha)}kgl$. The maps $f_{q+1}\mathbf{KGL} \to f_q\mathbf{KGL}$ are $\Sigma^{(q-1)(1+\alpha)}\beta'$, where β' is the arrow from the upper left corner to the lower right corner in the commutative diagram

$$\begin{tikzcd}
\Sigma^{1+\alpha}f_0\mathbf{KGL} \arrow[r, "\cong"] \arrow[d] & f_1\Sigma^{1+\alpha}\mathbf{KGL} \arrow[r, "\cong", "f_1\beta"'] \arrow[d] & f_1\mathbf{KGL} \arrow[d] \\
\Sigma^{1+\alpha}f_{-1}\mathbf{KGL} \arrow[r, "\cong"] & f_0\Sigma^{1+\alpha}\mathbf{KGL} \arrow[r, "\cong", "f_0\beta"'] & f_0\mathbf{KGL}
\end{tikzcd}$$

Combined with Lemma 1.5.6, and that $\mathbb{N}$ is cofinal in $(\mathbb{Z}, \leq)$ we get

$$\mathbf{KGL} \simeq \operatorname{hocolim}\left(kgl \xrightarrow{\Sigma^{-(1+\alpha)}\beta'} \Sigma^{-(1+\alpha)}kgl \xrightarrow{\Sigma^{-2(1+\alpha)}\beta'} \Sigma^{-2(1+\alpha)}kgl \to \cdots\right)$$

(that is, $\mathbf{KGL} = kgl[\beta'^{-1}]$). Since hocolim commutes with $\pi_\star$ (Lemma 1.1.16) this yields the colimit

$$\pi_\star\mathbf{KGL} = \pi_\star \operatorname{hocolim}_{q\geq 0}\left(\Sigma^{-q(1+\alpha)}kgl \xrightarrow{\Sigma^{-q(1+\alpha)}\beta'} \Sigma^{-(q+1)(1+\alpha)}kgl\right),$$
$$= \operatorname{colim}_{q\geq 0}\left(\Sigma^{-q(1+\alpha)}\pi_\star kgl \xrightarrow{v_1} \Sigma^{-(q+1)(1+\alpha)}\pi_\star kgl\right).$$

From the properties of colimits, since the maps $\pi_\star\Sigma^{-q(1+\alpha)}kgl \to \pi_\star\Sigma^{-(q+1)(1+\alpha)}kgl$ in the colimit are multiplication by $\pi_\star\beta' = v_1$, the canonical map $\pi_\star kgl \to \pi_\star\mathbf{KGL}$ fits in the exact sequence

$$0 \to \pi_\star kgl_{v_1} \to \pi_\star kgl \to \pi_\star\mathbf{KGL}.$$

When $\star = m + n\alpha, m \geq 0$, the maps

$$\pi_\star \Sigma^{-q(1+\alpha)} kgl \to \pi_\star \Sigma^{-(q+1)(1+\alpha)} kgl$$

are surjective, since $\pi_\star kgl$ is generated by v_1 and $\pi_{m+n\alpha} kgl$, $m \geq 0$ (this is clear from the computation of $\pi_\star kgl$, Chapter 4). Hence, we get the short exact sequence too. $\square$

Lemma 1.5.12. *We have an exact sequence*

$$0 \to \pi_\star ko_{w_1} \to \pi_\star ko \to \pi_\star \mathbf{KO},$$

where $\pi_\star ko_{w_1}$ denotes the w_1 torsion of $\pi_\star ko$. For $\star = m + n\alpha$, $m \geq 0$ and $m + n \geq 0$, this is in fact a short exact sequence

$$0 \to \pi_\star ko_{w_1} \to \pi_\star ko \to \pi_\star \mathbf{KO} \to 0.$$

Proof. The proof is similar to the proof of the previous lemma. Since $\mathbf{KO}$ is $4(1+\alpha)$-periodic we have

$$\Sigma^{4(1+\alpha)} \mathbf{KO} \xrightarrow[w]{\cong} \mathbf{KO}.$$

We apply f_{q+1} and Lemma 1.5.5, and get by induction $f_{4q}\mathbf{KO} = \Sigma^{4q(1+\alpha)} ko$. Under this identification the maps $f_{q+4} ko \to f_q ko$ are $\Sigma^{4q(1+\alpha)} w'$, where w' is the map from the upper left corner to the lower right corner in the commutative diagram

$$\begin{tikzcd}
\Sigma^{4(1+\alpha)} f_0 \mathbf{KO} \arrow[r, "\cong"] \arrow[d] & f_4 \Sigma^{1+\alpha} \mathbf{KO} \arrow[r, "\cong", "f_4 w"'] \arrow[d] & f_4 \mathbf{KO} \arrow[d] \\
\Sigma^{4(1+\alpha)} f_{-4} \mathbf{KO} \arrow[r, "\cong"] & f_0 \Sigma^{4(1+\alpha)} \mathbf{KO} \arrow[r, "\cong", "f_0 w"'] & f_0 \mathbf{KO}
\end{tikzcd}$$

Combined with Lemma 1.5.6, and that $4\mathbb{N}$ is cofinal we get

$$ko \simeq \operatorname{hocolim}_{q \geq 0} \left(\Sigma^{-4q(1+\alpha)} ko \xrightarrow{\Sigma^{-4q(1+\alpha)} w'} \Sigma^{-4(q+1)(1+\alpha)} ko \right)$$

(that is, $\mathbf{KO} = ko[w'^{-1}]$). Since hocolim commutes with $\pi_\star$ (Lemma 1.1.16) this yields the colimit

$$\pi_\star \mathbf{KO} = \operatorname{colim}_{q \geq 0} \left(\Sigma^{-4q(1+\alpha)} \pi_\star ko \xrightarrow{w_1} \Sigma^{-4(q+1)(1+\alpha)} \pi_\star ko \right).$$

The maps $\pi_\star ko \to \pi_\star ko$ in the colimit are multiplication by $w_1 = \pi_\star w'$. Hence, the canonical map $\pi_\star ko \to \pi_\star \mathbf{KO}$ fits into the exact sequence

$$0 \to \pi_\star ko_{w_1} \to \pi_\star ko \to \pi_\star \mathbf{KO},$$

where w_1 is the element corresponding to w'. When $\star = m + n\alpha, m \geq 0$, the maps

$$\pi_\star \Sigma^{-q(1+\alpha)} ko \to \pi_\star \Sigma^{-4(q+1)(1+\alpha)} ko$$

are surjective, since $\pi_\star ko$ is generated by w_1 and $\pi_{m+n\alpha} ko$, $m \geq 0$ (this is clear from the computation of $\pi_\star kgl$, Chapter 4). Hence, we get the short exact sequence too. $\square$

Remark 1.5.13. From the above lemma we see that if we have determined $\pi_\star kgl$ and its v_1-torsion, we have determined $\pi_{m+n\alpha}\mathbf{KGL}$ for $m \geq 0$. From the $(1+\alpha)$-periodicity we have in fact determined all of $\pi_\star \mathbf{KGL}$. There is an analogous statement for $\mathbf{KO}$.

Later we shall see that $\pi_\star kgl$ has no v_1-torsion (respectively $\pi_\star ko$ has no w_1-torsion), so $\pi_{m+n\alpha} kgl \to \pi_{m+n\alpha}\mathbf{KGL}$ (respectively $\pi_{m+n\alpha} ko \to \pi_{m+n\alpha}\mathbf{KO}$) is in fact an isomorphism for $m \geq 0$ and $m + n \geq 0$

1.6 Cellular Spectra

A particularly important class of spectra is cellular spectra. That is spectra which can be built up from the spheres $S^{m+n\alpha}$. In this section we define cellular spectra, give some of their properties and state which of the spectra we have encountered so far are cellular.

Definition 1.6.1 ([DI05])**.** Let $\mathcal{SH}(k)_{\text{cell}}$ be the smallest triangulated category such that

1. $\mathcal{SH}(k)_{\text{cell}}$ contains the spheres $S^{m+n\alpha}, m, n \in \mathbb{Z}$.
2. If $X \in \mathcal{SH}(k)_{\text{cell}}$ and $X \simeq Y$, then $Y \in \mathcal{SH}(k)_{\text{cell}}$.
3. $\mathcal{SH}(k)_{\text{cell}}$ is closed under hocolim.

An immediate consequences of this is that $\mathcal{SH}(k)_{\text{cell}}$ is closed under extensions and suspensions.

Lemma 1.6.2 ([DI05, Lemma 2.5])**.** *Let $X \to Y \to Z$ be a cofiber sequence of spectra. If two of the spectra are cellular, then so is the third spectrum.*

For cellular spectra stable equivalences are considerably easier to describe than the result of Lemma 1.1.8.

Proposition 1.6.3 ([DI05, Proposition 7.1])**.** *If $f : E \to F$ is a map between cellular spectra then f is a stable equivalence if an only if it induces isomorphisms on the homotopy groups $\pi_{m+n\alpha}E \to \pi_{m+n\alpha}F$ for all $m, n \in \mathbb{Z}$.*

Definition 1.6.4. A cell spectrum of finite type is a cellular spectrum X formed by iteratively forming cofibers

$$\bigvee_i S^{k_i+l_i\alpha} \to X_i \to X_{i+1}$$

and taking hocolim. Here $X_0 = *$, and there is a $k \in \mathbb{Z}$ such that there are no cells attached in dimension $m + n\alpha$ for $m < k$, and for each m there are only finitely many cells attached in dimension $m + n\alpha$.

Most of the spectra we have encountered so far are cellular.

- H is cellular of finite type. This is a consequence of [HKO11, Lemma 6] and the comment above Proposition 3.4.7 in Chapter 3.
- **MGL** is cellular [DI05, Theorem 6.2].
- **KGL** is cellular [DI05, Theorem 6.4].
- **KO** is cellular. This is a consequence of the results in [DI05] and the geometric models for **KO** in [ST13].
- The Moore spectra are cellular (they are a hocolim of spheres).

Lemma 1.6.5. *The functor f_0 preserves cellular spectra in the sense that if E is cellular then f_0E is cellular.*

Proof. We will prove that $f_0S^{m+n\alpha}$ is cellular for all $m, n \in \mathbb{Z}$. As a consequence, since f_0 commutes with hocolim, f_0 must preserve cellular objects.

From Lemma 1.5.5 we have that $f_0S^{m+n\alpha} = \Sigma^{m+n\alpha}f_{-n}S$. We will prove by induction that f_nS is cellular. To start the induction we have $f_nS = S, n \leq 0$, since $S \in \Sigma^{n\alpha}\mathcal{SH}^{\text{eff}}(k), n \leq 0$. Assume by induction that f_nS is cellular. The cofiber sequence in Equation (3.4) is

$$f_{n+1}S \to f_nS \to s_nS.$$

From [RSØ14, Theorem 2.6] we have that s_nS is cellular for all n. Hence, by Lemma 1.6.2 $f_{n+1}S$ must be cellular. $\square$

1.7 The Motivic Steenrod Algebra

The results in this section are true over perfect fields with characteristic not equal to 2.

The motivic Steenrod algebra mod 2 over a field k is the bigraded algebra of bistable operations on the mod 2 cohomology of smooth schemes over k. This algebra is denoted by $A^\star$. A bistable operation in degree $m + n\alpha$ on mod 2-cohomology is a collection of natural transformations

$$\phi_{p+q\alpha} : H^{p+q\alpha}(-) \to H^{p+m+(q+n)\alpha}(-),$$

which commute with suspension with both spheres (recall Equation (1.1) and Section 1.2), i.e., we have commutative diagrams:

$$\begin{tikzcd} H^{p+q\alpha}(X) \arrow[r, "\phi_{p+q\alpha}"] \arrow[d, "\sigma_s"] & H^{p+m+(q+n)\alpha}(X) \arrow[d, "\sigma_s"] \\ H^{p+1+q\alpha}(X \wedge S^1_s) \arrow[r, "\phi_{p+1+q\alpha}"] & H^{p+m+1+(q+n)\alpha}(X \wedge S^1_s) \end{tikzcd}$$

$$\begin{tikzcd} H^{p+q\alpha}(X) \arrow[r, "\phi_{p+q\alpha}"] \arrow[d, "\sigma_t"] & H^{p+m+(q+n)\alpha}(X) \arrow[d, "\sigma_t"] \\ H^{p+(q+1)\alpha}(X \wedge S^1_t) \arrow[r, "\phi_{p+(q+1)\alpha}"] & H^{p+m+(q+1+n)\alpha}(X \wedge S^1_t) \end{tikzcd}$$

The algebra structure is given by addition of operations and composition, making it an algebra over $\mathbb{Z}/2$. Henceforth, we only refer to the motivic Steenrod algebra mod 2 as the Steenrod algebra, or simply $A^\star$. The field which we work over is implicit.

The description of the Steenrod algebra over perfect fields of characteristic 0 is known from [Voe03] and [Voe10]. In [HKØ13] the description is extended to perfect fields of nonzero characteristic.

The simplest operations in $A^\star$ are those which are induced by the smash-product of elements $u \in H^{m+n\alpha}(S)$

$$H^{p+q\alpha}(X) \xrightarrow{-\wedge u} H^{p+m+(q+n)\alpha}(X).$$

More elaborate operations are the motivic Steenrod operations mod 2. They are denoted by Sq^i, and are of bidegree $\lceil i/2 \rceil + \lfloor i/2 \rfloor \alpha$. Their definition is found in [Voe03, Section 9] and [HKØ13, Section 2.4]. These operations satisfy analogous properties to the topological power operations, and are subject to a motivic version of the Adem relations (see [Voe03, Section 10] or [HKØ13, Theorem 5.1] for the actual equations). Note that $\mathrm{Sq}^0 = \mathrm{id}$, and $\mathrm{Sq}^1 = \beta$, the Bockstein homomorphism. The operations mentioned above generate all the bistable operations, which is why we call $A^\star$ the Steenrod algebra.

Proposition 1.7.1 ([Voe10, Theorem 3.49], [HKØ13, Theorem 1.1])**.** *The motivic Steenrod algebra is generated by $H^\star$ and the motivic Steenrod operations mod 2.*

The Steenrod algebra is intimately related to motivic cohomology. In fact we have the following isomorphisms.

Theorem 1.7.2 ([Voe03, Proposition 2.7], [HKØ13, Theorem 1.1])**.** *An element of $H^\star H$ is a map $H \to \Sigma^{m+n\alpha} H$. Applying $[X, -]$ we readily get a collection of bistable operations on cohomology of bidegree $m + n\alpha$. Hence, we have a map $H^\star H \to A^\star$. This map is an isomorphism, that is*

$$H^\star H \cong A^\star.$$

Define admissible polynomials to be polynomials in the Sq^i of the form $\mathrm{Sq}^I := \mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_n} \cdots$, where $I = (i_1, i_2, i_3, \ldots, i_n, \ldots)$ such that $i_n \geq 2i_{n+1}$. The admissible polynomials form a basis over $H^\star$, that is, they generate $A^\star$ as a left $H^\star$-module and are linearly independent over $H^\star$ [Voe03, Lemma 11.1, Corollary 11.5]. As an algebra over $H^\star$, the Steenrod algebra is generated by the indecomposable elements Sq^{2^i}. From this description it is clear that the Steenrod algebra is finite in each degree, both as a $H^\star$-module and $\mathbb{Z}/2$-module. Consequently, the same holds for its dual, and dualizing twice bring us back to $A^\star$ via the canonical isomorphism. It is possible to equip $A^\star$ with a map $A^\star \to A^\star \otimes_{H^\star} A^\star$. This map factors through a submodule of $A^\star \otimes_{H^\star} A^\star$ which is a ring compatible with the product on $A^\star \otimes_{\mathbb{Z}/2} A^\star$. This is done in [Voe03, Section 11]. This "coalgebra"-map is associative and cocommutative and will provide the dual with the structure of a commutative Hopf algebroid.

The dual of the Steenrod algebra has a simple description. The dual is defined as

$$A_\star := \mathrm{Hom}^{-\star}_{H^\star}(A^\star, H^\star).$$

Here $\mathrm{Hom}^{-\star}$ are the negatively graded hom groups (Appendix A). We use the negative grading on the hom groups so $A_\star$ becomes a module over $H_\star$, because of the following isomorphism of $H^\star$-modules,

$$H_\star = H^{-\star} \cong \mathrm{Hom}^{-\star}_{H^\star}(H^\star, H^\star).$$

The structure of the dual is described in the following proposition.

Proposition 1.7.3 ([HKØ13, p. 35], [Voe03, Remark 12.12])**.** *The dual Steenrod algebra is a commutative $H_\star$-algebra:*

$$A_\star \cong H_\star[\tau_0, \tau_1, \ldots][\xi_1, \xi_2, \ldots]/(\tau_i^2 + \tau\xi_{i+1} + \rho(\tau_{i+1} + \tau_0\xi_{i+1})).$$

Here τ and ρ are the elements of $H_\star$ in degree $1-\alpha$ and $-\alpha$ respectively, the duals of the elements defined in Proposition 1.2.5. The degrees of τ_i and ξ_i are $(2^i-1)(1+\alpha)+1$ and $(2^i-1)(1+\alpha)$ respectively. Furthermore, $(H_\star, A_\star)$ is a bigraded Hopf algebroid. The structure maps are defined by

$$\eta_L\tau = \tau, \quad \eta_R\tau = \tau + \rho\tau_0,$$
$$\epsilon\tau_i = 0, i \geq 0, \quad \epsilon\xi_i = 0, i > 0$$
$$\Delta\xi_k = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i,$$
$$\Delta\tau_k = \tau_k \otimes 1 + \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \tau_i.$$

Note the convention $\xi_0 = 1$ in the above sums. This makes $A_\star$ into a connected Hopf-algebroid. An element $x \in H_\star \subset A_\star$ is mapped by Δ to $x \otimes 1$.

Remark 1.7.4. Since $A^\star$ is free as a left $H^\star$-module, $A_\star$ is free as a left $H_\star$-module. Therefore $A_\star$ is a flat left $H_\star$-module. Hence, we are free to apply the machinery of Appendix A to the Hopf algebroid $(H_\star, A_\star)$. Furthermore, $H_\star H \cong A_\star$ ([HKØ13, Proposition 5.3]).

Definition 1.7.5. Let Q_i denote the dual of τ_i (that is, the element of $A^\star$ corresponding to the $H_\star$-linear map $A_\star \to \Sigma^{|\tau_i|}H_\star$, which takes the value 1 on τ_i and zero on all other basis elements of $A_\star$). The Q_i are called the Milnor primitives.

The first two Milnor primitives are $Q_0 = \mathrm{Sq}^1$ and $Q_1 = \mathrm{Sq}^3 + \mathrm{Sq}^2\,\mathrm{Sq}^1$ ([Voe03, Proposition 12.4, Proposition 13.6]). In contrast to topology, the Milnor primitives cannot be constructed inductively with the formula $Q_k = [\mathrm{Sq}^{2^k}, Q_{k-1}]$.

We will make use of three particular subalgebras of the Steenrod algebra in the computations. They are $E(0)$, $E(1)$ and $A(1)$ in the following more general definition.

Definition 1.7.6. Let $E(n)$ and $A(n)$ be subalgebras of $A^\star$ on the generators

$$E(n) := \langle Q_i, i \leq n \rangle,$$
$$A(n) := \langle \mathrm{Sq}^{2^i}, i \leq n \rangle.$$

For all n, $E(n)$ is an exterior algebra and a subalgebra of $A(n)$ ([Voe03, Proposition 13.4, Proposition 13.6]).

Using this we compute the duals of the subalgebras $E^\star(n)$ and $A^\star(n)$ in Definition 1.7.6. They are Hopf algebroids.

Proposition 1.7.7 ([Hil11, p. 2], [Gre12, 3])**.**

$$\begin{aligned}
E_\star(n) &:= \operatorname{Hom}^{-\star}_{H^\star}(E(n), H^\star) \cong (H_\star, H_\star[\tau_0, \dots, \tau_n]/(\tau_i^2 + \rho\tau_{i+1}, \tau_n^2)),\\
A_\star(n) &:= \operatorname{Hom}^{-\star}_{H^\star}(A(n), H^\star)\\
&\cong (H_\star, H_\star[\xi_1, \dots, \xi_n][\tau_0, \dots, \tau_n]/(\xi_i^{2^{n-i+1}}, \tau_i^2 + \rho\tau_{i+1} + (\rho\tau_0 + \tau)\xi_{i+1}))
\end{aligned}$$

(the ideals we take the quotient by are generated by all i for which the expressions make sense, with the convention that $\tau_i = 0, i \geq n$ and $\xi_i = 0$ for $i > n$). The inclusions $E(n) \hookrightarrow A(n)$ dualize to surjections $A_\star(n) \twoheadrightarrow E_\star(n)$, where $\tau_i \mapsto \tau_i$ and $\xi_i \mapsto 0$.

Let $B \subset A^\star$ be a subalgebra and subcoalgebra of $A^\star$ with the coproduct defined above. Then we define the quotient $A^\star//B := A^\star \otimes_B H^\star$. Here $H^\star$ is equipped with the left B-module structure induced by the left $A^\star$-module structure, and $A^\star$ has the right B-module structure induced by the inclusion $B \hookrightarrow A^\star$ and the algebra structure. The following lemma is very useful for computations.

Lemma 1.7.8. *We have the following isomorphism of $H^\star$-modules*

$$A^\star//B \cong A^\star/A^\star \cdot IB,$$

where $IB := \ker(B = B \otimes_{H^\star} H^\star \to H^\star)$.

Proof. By definition, $A_\star \otimes_B H^\star$ is the cokernel of the diagram

$$A^\star \otimes B \otimes H^\star \to A^\star \otimes H^\star,$$

where the map is the difference of the module maps. Consider an element $a \otimes b \otimes h \in A^\star \otimes B \otimes H^\star$. This is mapped to

$$a \otimes b \otimes h \mapsto ab \otimes h - a \otimes (b \cdot h) = \begin{cases} abh & b \in IB, \\ 0 & b \notin IB \end{cases}$$

(the identification $A^\star \otimes H^\star \cong H^\star$ is implicit). Hence, the image of this map is $A^\star \cdot IB$. □

Remark 1.7.9. Observe that $IE(n)$ and $IA(n)$ are generated as $H^\star$-modules by all the generators of $E(n)$ or $A(n)$ except 1. This will be very useful when computing with quotients by $E(n)$ and $A(n)$ later.

The following lemma will be important for spectra X such that $\operatorname{Hom}^{-\star}(H^\star X, H^\star) = H_\star X$, and the cohomology $H^\star X$ is a quotient $A^\star//B$ for some B. This is true for spectra X satisfying some finiteness conditions, see Remark 4.1.1.

Lemma 1.7.10. *Let B be a subalgebra and subcoalgebra of $A^\star$, free over $H^\star$, and let $D = A//B$. Let $B_\star := \operatorname{Hom}^{-\star}_{H^\star}(B, H^\star)$ and $D_\star := \operatorname{Hom}^{-\star}_{H^\star}(D, H^\star)$ be their duals. Then*

$$D_\star \cong A_\star \square_{B_\star} H_\star.$$

(the cotensor product $\square$ is defined in Definition A.1.12)

Proof. This is a consequence of

$$D = \operatorname{coker}(A^\star \otimes B \otimes H^\star \to A^\star \otimes H^\star), \qquad A_\star \square_{B_\star} H_\star = \ker(A_\star \otimes H_\star \to A_\star \otimes B \otimes H_\star),$$

and that $\operatorname{Hom}(-, H^\star)$ is left exact. □

When we compute the cohomology of some of the spectra we have considered we will need some short exact sequences of quotients of $A^\star$-modules. Before we can prove this we need the following technical lemma.

Lemma 1.7.11 ([Gre12, Lemma 3.2.15])**.** *The Steenrod algebra $A^\star$ is free as a left $A(n)$-module.*

Remark 1.7.12. The proof of the above lemma is for fields of characteristic 0. But from the description of the Steenrod algebra for perfect fields with positive characteristic, the comodule structures are the same. Hence, the proof is identical.

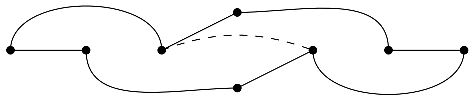
Lemma 1.7.13. *There are short exact sequences of left $H^\star$-modules:*

$$0 \longrightarrow \Sigma A//E(0) \xrightarrow{\cdot \mathrm{Sq}^1} A \longrightarrow A//E(0) \longrightarrow 0$$

$$0 \longrightarrow \Sigma^{2+\alpha} A//E(1) \xrightarrow{\cdot \mathrm{Q}_1} A//E(0) \longrightarrow A//E(1) \longrightarrow 0$$

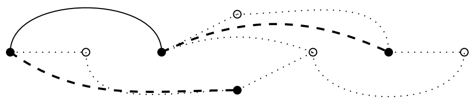
$$0 \longrightarrow \Sigma^{1+\alpha} A//A(1) \xrightarrow{\cdot \mathrm{Sq}^2} A//E(1) \longrightarrow A//A(1) \longrightarrow 0$$

Proof. Note that $E(0), E(1)$ and $A(1)$ are all subalgebras and subcoalgebras of $A(1)$. Since $A^\star$ is free as an $A(1)$-module, we might as well prove the analogous short exact sequences for $A(1)$ and then tensor with $A^\star$ to prove the lemma. This has a computational advantage, since $A(1)$ has only eight generators over $H^\star$. The multiplicative structure of the generators of $A(1)$ is displayed in Figure 1.7. Only nonzero multiplications are shown. Straight lines are right multiplication by Sq^1. The curves are right multiplication by Sq^2. The dashed curve indicates that right multiplication by Sq^2 is τ times the generator in that degree, that is $\mathrm{Sq}^2\mathrm{Sq}^2 = \tau\,\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1$. The generators are ordered by total degree from left to right.



(1.7)

The corresponding picture for $A(1)//E(0) = A(1)/(\mathrm{Sq}^1)$ is displayed in Figure 1.8. The generators in $A(1)$ which are zero in $A(1)//E(0)$ are drawn as circles. The multiplications which are now zero are dotted. Multiplication by Sq^3 is drawn with thick dashed lines.



(1.8)

The analogous pictures for $A(1)//E(1)$ and $A(1)//A(1)$ are even simpler. They have the generators $\{1, \mathrm{Sq}^2\}$ and $\{1\}$, respectively. It is then straightforward to check that the sequences are exact.

To prove exactness of the first sequence, we project the picture of $A(1)$ onto the picture of $A(1)//E(0)$. All the dots corresponding to hollow dots are mapped to zero. From the picture of $A(1)//E(0)$ we see that they are exactly the image of right multiplication by Sq^1. Hence, the first sequence is exact.

For the second sequence we follow the same procedure. Since $\mathrm{Q}_1 = \mathrm{Sq}^3$ modulo (Sq^1) we can multiply by $\mathrm{Sq}^3 = \mathrm{Sq}^1\mathrm{Sq}^2$ in place of Q_1. We see that the image of right multiplication by Sq^3 is exactly what is mapped to zero, when projecting $A//E(0) \twoheadrightarrow A//E(1)$. The same argument is sufficient to prove exactness of the third sequence. $\square$

Remark 1.7.14. In the last sections we have seen several properties which seems to be analogous to phenomena in topology. This percieved likeness can be made precise by topological realization. Given a field k with an embedding $k \hookrightarrow \mathbb{C}$ there is a functor $\mathcal{SH}(k) \to \mathcal{SH}^{\mathrm{top}}$. Cofiber sequences are preserved under this map. This induces a map on homotopy groups, such that an element in degree $m + n\alpha$ is mapped to degree $m + n$. The motivic Eilienberg-Maclane spectra map to topological Eilenberg-Maclane spectra, and the spectra for algebraic and Hermitian K-theory map to complex and real K-theory. The induced map on motivic cohomology maps τ to 1.

This correspondence is a useful safety check when we work with motivic homotopy theory. Any statement in the stable motivic homotopy category should be true under topological realization. It can also be used to pull results in topology back to the motivic world.

1.8 Cohomology of $H\mathbb{Z}_{(2)}$, ko and kgl

In this section we compute the cohomology groups of ko. On our way we compute the cohomology of $H\mathbb{Z}_{(2)}$ and kgl. As a corollary we get the homology groups of ko because of Lemma 1.7.10. The structure of the arguments are the same as in [IS11, 5.3], and we give citations to the corresponding lemmas. However, the proofs are sometimes substantially different, and we supply more details. Throughout this section motivic cohomology refer to motivic cohomology with $\mathbb{Z}/2$-coefficients.

The main goal is the cohomology of ko. There are two paths leading to this. One possibility is to use Proposition 1.5.8 and Theorem 1.3.11, hence, we can immideately skip to the computation of the cohomology of ko. The second option is to use Proposition 1.5.8 and Lemma 1.3.13 to first compute the cohomology of $H\mathbb{Z}_{(2)}$ and kgl. We choose the last option, although the computations are essentially merely special cases of the proof of Theorem 1.3.11. The cofiber sequence from Lemma 1.3.13 can be obtained from the cofiber sequence for the zero slice of **KGL** (Equation (3.4)). If we knew more about $H^\star kgl$, specifically that $H^{2+\alpha}kgl = 0$, we could avoid using Proposition 1.5.8. Unfortunately we do not know of an alternative proof of this.

Cohomology of $H\mathbb{Z}_{(2)}$

Consider the map $H\mathbb{Z}_{(2)} \xrightarrow{2} H\mathbb{Z}_{(2)}$ with cofiber $H\mathbb{Z}_{(2)} \xrightarrow{2} H\mathbb{Z}_{(2)} \xrightarrow{p} C_2 \xrightarrow{\delta} \Sigma H\mathbb{Z}_{(2)}$.

Lemma 1.8.1 ([IS11, Lemma 5.3])**.** *The cofiber C_2 is homotopic to H.*

Proof. Consider the map of cofiber sequences

$$\begin{array}{ccccccc} H\mathbb{Z} & \xrightarrow{2} & H\mathbb{Z} & \longrightarrow & H & \longrightarrow & \Sigma H\mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H\mathbb{Z}_{(2)} & \xrightarrow{2} & H\mathbb{Z}_{(2)} & \longrightarrow & C_2 & \longrightarrow & \Sigma H\mathbb{Z}_{(2)} \end{array}$$

where the two leftmost vertical arrows are the ones induced by $\mathbb{Z} \to \mathbb{Z}_{(2)}$. If we consider the long exact sequence of homotopy group and narrow down to the map of short exact sequences around $\pi_\star H \to \pi_\star C_2$ we get:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{im}(\pi_\star H\mathbb{Z} \to \pi_\star H) & \longrightarrow & \pi_\star H & \longrightarrow & \ker(\pi_{\star-1} H\mathbb{Z} \xrightarrow{2} \pi_{\star-1} H\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{im}(\pi_\star H\mathbb{Z}_{(2)} \to \pi_\star C_2) & \longrightarrow & \pi_\star C_2 & \longrightarrow & \ker(\pi_{\star-1} H\mathbb{Z}_{(2)} \xrightarrow{2} \pi_{\star-1} H\mathbb{Z}_{(2)}) & \longrightarrow & 0 \end{array}$$

From Lemma 1.4.4 we have $\pi_\star H\mathbb{Z} \otimes \mathbb{Z}_{(2)} = \pi_\star H\mathbb{Z}_{(2)}$. Since $\mathbb{Z}_{(2)}$ is flat we have isomorphisms

$$\begin{aligned} \mathrm{im}(\pi_\star H\mathbb{Z} \to \pi_\star H) &\cong (\pi_\star H\mathbb{Z})/2(\pi_\star H\mathbb{Z}) \\ &\cong (\pi_\star H\mathbb{Z} \otimes \mathbb{Z}_{(2)})/(2\pi_\star H\mathbb{Z} \otimes \mathbb{Z}_{(2)}) \cong \mathrm{im}(\pi_\star H\mathbb{Z}_{(2)} \to \pi_\star C_2), \end{aligned}$$

and

$$\begin{aligned} \ker(\pi_\star H\mathbb{Z} \xrightarrow{2} \pi_\star H\mathbb{Z}) &\cong \mathrm{Tor}_1^{\mathbb{Z}}(\pi_\star H\mathbb{Z}, \mathbb{Z}/2) \cong \mathrm{Tor}_1^{\mathbb{Z}}(\pi_\star H\mathbb{Z}, \mathbb{Z}/2 \otimes \mathbb{Z}_{(2)}) \\ &\cong \mathrm{Tor}_1^{\mathbb{Z}}(\pi_\star H\mathbb{Z} \otimes \mathbb{Z}_{(2)}, \mathbb{Z}/2) \cong \ker(\pi_\star H\mathbb{Z} \otimes \mathbb{Z}_{(2)} \xrightarrow{2} \pi_\star H\mathbb{Z} \otimes \mathbb{Z}_{(2)}). \end{aligned}$$

Hence, the maps of kernels and images are isomorphisms (this is also straightforward, albeit tedious, to check by hand). The snake lemma then implies that the middle map is an isomorphism. Since both H and C_2 are cellular, Proposition 1.6.3 implies $H \simeq C_2$. $\square$

Recall that $H^\star H = A^\star$, the motivic Steenrod algebra.

Lemma 1.8.2 ([IS11, Lemma 5.4])**.** *The map $\delta^\star p^\star$ is right multiplication by* $\mathrm{Sq}^1 = Q_0$.

Proof. Consider the long exact sequence in motivic cohomology induced by the cofiber sequence

$$H\mathbb{Z}_{(2)} \xrightarrow{2} H\mathbb{Z}_{(2)} \xrightarrow{p} C_2 \xrightarrow{\delta} \Sigma H\mathbb{Z}_{(2)}.$$

Since $2^\star = 0$ in motivic cohomology with mod 2 coefficients, it splits into short exact sequences

$$0 \to H^\star \Sigma H\mathbb{Z}_{(2)} \xrightarrow{\delta^\star} H^\star H \xrightarrow{p^\star} H^\star H\mathbb{Z}_{(2)} \to 0.$$

Considering degrees we must have $\delta^\star p^\star 1 = \mathrm{Sq}^1$. Since $\delta^\star p^\star$ is an $A^\star$-module map we have proved the lemma. $\square$

Proposition 1.8.3. *Motivic cohomology of $H\mathbb{Z}_{(2)}$ is equal to $A//E(0)$.*

Proof. We have $0 = p^\star \delta^\star p^\star = p^\star(- \cdot \mathrm{Sq}^1)$, hence, $p^\star$ annihilates the left ideal generated by Sq^1, and extends to a map $\bar{p}^\star : H^\star H\mathbb{Z}_{(2)} \to A^\star//E(0)$. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & H^\star \Sigma H\mathbb{Z}_{(2)} \arrow[r, "\delta^\star"] \arrow[d, "\Sigma\bar{p}^\star"] & H^\star H \arrow[r, "p^\star"] \arrow[d, "\cong"] & H^\star \mathbb{Z}_{(2)} \arrow[r] \arrow[d, "\bar{p}^\star"] & 0 \\
0 \arrow[r] & \Sigma A//E(0) \arrow[r] & A \arrow[r] & A//E(0) \arrow[r] & 0
\end{tikzcd}$$

Here the top row is exact by the long exact sequence in cohomology. The bottom row is exact by Lemma 1.7.13. The right square is commutative by the definition of $\bar{p}^\star$. The left square is commutative by Lemma 1.8.2 above. In more detail, an element $x \in H^\star \Sigma H\mathbb{Z}_{(2)}$ is the image of some $y \in H^\star \Sigma H$, $p^\star(y) = x$. Then $\delta^\star(x) = \delta^\star p^\star x = Sq^1 x$ (the isomorphism between $H^\star$ and $A^\star$ is implicit). The snake lemma and induction imply that $\bar{p}^\star$ is an isomorphism. $\square$

Cohomology of kgl

We proceed to compute the cohomology of kgl. The computation is essentially a special case of [Orm11, Theorem 3.8] and [Wil75]. However, since we are working in positive characteristic we need results of [Hoy13].

Recall the cofiber sequence in Equation (1.9),

$$\Sigma^{1+\alpha} kgl \xrightarrow{\beta} kgl \xrightarrow{p} H\mathbb{Z}_{(2)} \xrightarrow{\delta} \Sigma^{2+\alpha} kgl. \tag{1.9}$$

Lemma 1.8.4 ([IS11, Lemma 5.6])**.** *The composition $\delta^\star p^\star$ is equal to right multiplication by Q_1, a Milnor primitive.*

Proof. Consider the long exact sequence in cohomology of the above cofiber sequence,

$$H^{-2-\alpha} \xrightarrow{\delta^\star} H^0 H\mathbb{Z}_{(2)} \xrightarrow{p^\star} H^0 kgl \xrightarrow{\beta^\star} H^{-1-\alpha} kgl.$$

From Corollary 1.5.9 and Theorem 1.3.11 the first and last group are zero, and $p^\star(1) = 1$. Similarly we have

$$H^{1+\alpha} kgl \xrightarrow{\beta^\star} H^0 \Sigma^{2+\alpha} kgl \xrightarrow{\delta^\star} H^{2+\alpha} H\mathbb{Z}_{(2)} \to H^{2+\alpha} kgl.$$

For $\star = 2 + \alpha$, Corollary 1.5.9 together with Theorem 1.3.11 imply that the last group is zero, and $\delta^\star(1) = Q_1$, the only nonzero element in bidegree $2 + \alpha$. $\square$

Proposition 1.8.5 ([IS11, Theorem 5.7]). *Motivic cohomology of kgl is equal to $A//E(1)$.*

Proof. We have $0 = p^\star\delta^\star p^\star = p^\star(- \cdot Q_1)$, hence, $p^\star$ annihilates the left ideal generated by Q_1 in $A//E(0)$, and $p^\star$ extends to a map $\bar{p}^\star : H^\star kgl \to A//E(1)$. Consider the diagram

$$\begin{tikzcd}
H^\star\Sigma kgl \arrow[r, "\Sigma\beta^\star"] & H^\star\Sigma^{2+\alpha} \arrow[r, "(\delta')^\star"] \arrow[d, "\Sigma^{2+\alpha}\bar{p}^\star"] & H^\star H\mathbb{Z}_{(2)} \arrow[r, "p^\star"] \arrow[d, "\cong"] & H^\star kgl \arrow[r, "\beta^\star"] \arrow[d, "\bar{p}^\star"] & H^\star\Sigma^{1+\alpha}kgl \\
0 \arrow[r] & \Sigma^{2+\alpha}A//E(1) \arrow[r] & A//E(0) \arrow[r] & A//E(1) \arrow[r] & 0
\end{tikzcd}$$

The top row is exact by the long exact sequence in cohomology. The bottom row is exact by Lemma 1.7.13. The right square commutes by definition. The left square commutes by Lemma 1.8.4 and a simililar argument as in the proof of Proposition 1.8.3.

The proof that $\bar{p}^\star$ is an isomorphism is by induction on the total degree $m+n$ of the bidegree $m+n\alpha$. When $m+n < 0$, the groups are zero and we trivially have an isomorphism. Assume that $\bar{p}^\star$ is an isomorphism for $m+n < l$. Then $\Sigma^{2+\alpha}\bar{p}^\star$ is an isomorphism in the degrees $m+n < l+3$. Hence, $\delta^\star$ is injective and $\Sigma\beta^\star$ is zero for $m + n < l + 3$. Moving this information to the right hand part of the diagram we get that $\beta^\star$ is zero for $m + n < l + 2$, hence, by the snake lemma $\bar{p}^\star$ is an isomorphism for for $m + n < l + 2$. By induction $\bar{p}^\star$ is an isomorphism. □

Cohomology of *ko*

Finally we can compute the cohomology of ko.

Lemma 1.8.6 ([IS11, Lemma 5.10]). *The composition $\delta^\star p^\star$ is equal to right multiplication by* Sq^2.

Proof. We take cohomology of the cofiber sequences in Equation (1.5),

$$\begin{tikzcd}
H^\star\Sigma^{1+\alpha}S \arrow[r, "\delta^\star"] & H^\star C\eta \arrow[r, "(p')^\star"] & H^\star S \\
H^\star\Sigma^{1+\alpha}ko \arrow[r, "\delta^\star"] \arrow[u] & H^\star kgl \arrow[r, "p^\star"] \arrow[u] & H^\star S \arrow[u]
\end{tikzcd}$$

The module $H^\star C\eta$ is a free $H^\star$-module on two generators x and y in degree 0 and $1 + \alpha$. The proof of this is similar to Lemma 1.4.2, if one remember that the maps are $H^\star$-linear. The cone $C\eta$ is a suspension spectrum of $\mathbb{P}^2$, which implies that $\mathrm{Sq}^2 x = y$ [IS11, Lemma 5.10]. Since $S \to C\eta \to kgl \to S$ is the identity on S this implies that $1 \in H^\star kgl$ maps to $x \in C\eta$. Then $H^\star H$-lineraity implies that Sq^2 maps to y. We then chase the diagram to obtain $\delta^\star p^\star(1) = \mathrm{Sq}^2$. This is because $p^\star(1) \mapsto 1 \in H^0 S$, which $(\delta')^\star$ map to $y \in H^{1+\alpha}C\eta$. From commutativity of the diagram $\delta^\star p^\star(1)$ must be nonzero, that is Sq^2, the only nonzero element in bidegree $1 + \alpha$. □

Proposition 1.8.7 ([IS11, Theorem 5.11]). *Motivic cohomology of ko is equal to $A//A(1)$.*

Proof. As in the proof of the previous two theorems, $p^\star$ extends to a map $\bar{p}^\star : H^\star ko \to A//A(1)$ Consider the diagram

$$\begin{tikzcd}
H^\star\Sigma ko \arrow[r, "\Sigma\eta^\star"] & H^\star\Sigma^{1+\alpha}ko \arrow[r, "\delta^\star"] \arrow[d, "\Sigma^{1+\alpha}\bar{p}^\star"] & H^\star kgl \arrow[r, "p^\star"] \arrow[d, "\cong"] & H^\star ko \arrow[r, "\eta^\star"] \arrow[d, "\bar{p}^\star"] & H^\star\Sigma^{\alpha}ko \\
0 \arrow[r] & \Sigma^{1+\alpha}A//A(1) \arrow[r] & A//E(1) \arrow[r] & A//A(1) \arrow[r] & 0
\end{tikzcd}$$

As before, both rows are exact, the right square is commutative by definition and the left square by Lemma 1.8.6. The proof that $\bar{p}^\star$ is an isomorphism is the same as in Proposition 1.8.5. We do induction on the total degree $m + n$ of the bidegree $m + n\alpha$. When $m + n \ll 0$ (a positive cell structure on ko implies this, cf. Remark 4.1.1), the groups are zero and $\bar{p}^\star$ is trivially an isomorphism. Assume that $\bar{p}^\star$ is an isomorphism for $m+n < l$. Then $\Sigma^{1+\alpha}\bar{p}^\star$ is an isomorphism for $m + n < l + 2$, hence $\delta^\star$ is injective and $\Sigma\eta^\star$ is zero in the same range. Then $\eta^\star$ is zero for $m + n < l + 1$, and $\bar{p}^\star$ is an isomorphism for $m + n < l + 1$. □

With the above results on the cohomology we obtain the homology by a simple dualization lemma, Lemma 1.7.10. Application of the lemma yields

$$\begin{aligned} H_\star \mathbb{Z}_{(2)} &= A_\star \,\square_{E_\star(0)}\, H_\star, \\ H_\star ko &= A_\star \,\square_{E_\star(1)}\, H_\star, \\ H_\star kgl &= A_\star \,\square_{A_\star(1)}\, H_\star. \end{aligned}$$

Here $E_\star(0)$, $E_\star(1)$ and $A_\star(1)$ are the Hopf algebroids defined in Proposition 1.7.7.

2 K-theory

2.1 Milnor K-theory

In this section we define Milnor K-theory of fields and state some of its properties for fields of particular interest to us.

Definition 2.1.1. Let k be a field. Milnor K-theory $K_*^M(k)$ of a field is defined as the quotient of the tensor algebra

$$T(k^\times) = \mathbb{Z} \oplus k^\times \oplus (k^\times \otimes k^\times) \oplus (k^\times \otimes k^\times \otimes k^\times) \oplus \cdots$$

by the ideal generated by $l(x) \otimes l(x-1), x \ni \{0,1\}$, where l is the canonical map $k^\times \to T(k^\times)$. Note that $k^\times$ is written as an additive group above.

Define the Steinberg-symbols $\{x_1, \dots, x_n\} := l(x_1) \otimes \cdots \otimes l(x_n)$. Then $\{x_1, \cdots, x_n\} = 0$ if $x_i + x_{i+1} = 1$ for some i. Note that

$$\{x, -x\} = \{x, (1-x)(1-x^{-1})^{-1}\} = \{x, 1-x\} - \{x, 1-x^{-1}\} = 0.$$

Hence, $\{x, y\} = -\{y, x\}$ (this can be generalized to symbols of arbitrary length and arbitrary permutations, but we do not need this). We will only be concerned with Milnor K-theory of finite fields, which have a rather simple structure.

Proposition 2.1.2. *For finite fields*

$$K_i^M(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{F}_q^\times & i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let x be a generator of $\mathbb{F}_q^\times$. Then the symbols of the form $\{x, \dots, x\}$ generate all the groups in degree ≥ 1. We prove that they are zero in degree ≥ 2. If the order q is even, then $\{x, x\} = \{x, -x\} = 0$ by the calculation above. For q odd, $2\{x, x\} = 0$ and for m and n odd

$$\{x, x\} = mn\{x, x\} = \{x^m, x^n\},$$

by antisymmetry. An element $u \in \mathbb{F}_q^\times$ is a non-square if and only if $u = x^m$, for m odd. The map $u \mapsto 1 - u$ is an involution on $\mathbb{F}_q - \{0, 1\}$. This set has $(q-2)$ elements, $(q-1)/2$ non-squares and $(q-3)/2$ squares (Proposition B.1.2), so there exists a non-square u such that $1 - u$ is a non-square. That is, there exist odd m and n such that $x^m = u, x^n = 1 - u$. It follows that $\{x, x\} = mn\{x, x\} = \{u, 1-u\} = 0$. $\square$

We also need to know that Milnor K-theory of an algebraically closed field is divisible.

Proposition 2.1.3. *Let $k = \bar{k}$ be an algebraically closed field. Then $K_m^M(k)$ is a divisible group for $m \geq 1$.*

Proof. Observe that $k^\times$ is divisible, since $x^n = a$ has a solution for every $a \in k^\times$. Trivially all tensor products $k^\times \otimes \cdots \otimes k^\times$ are divisible. Hence, the groups $K_m^M(k)$ are divisible for $m \geq 1$, since they are quotients of divisible groups (this can be seen by application of the snake lemma). $\square$

In fact, for algebraically closed fields, $K_*^M(k)$ is uniquely divisible ([Wei13, III. Exercise 7.3]), however, we will not need this. In general the Milnor K-theory of a field is extremely complicated (nontrivial divisible groups are necessarily infinitely generated).

We will primarily be concerned with Milnor K-theory reduced modulo 2.

Definition 2.1.4. Milnor K-theory reduced modulo 2 is defined as the quotient

$$k_*^M(k) := K_*^M(k)/2K_*^M(k).$$

In particular, in degree 1 we have $k^\times/(k^\times)^2$.

We have $k_*^M(\mathbb{F}_q) = \Lambda_{\mathbb{Z}/2}(x), |x| = 1$, since every element of $\mathbb{F}_q^\times$ is either a square or a non-square. For algebraically closed fields, by the divisibility proposition above, $k_*^M(\bar{k}) = \mathbb{Z}/2$ concentrated in degree 0.

2.2 Hermitian K-theory

In this section we define Hermitian K-theory of fields. We sketch the definition of the hermitian K-groups $K_i^h, i = 0, 1, 2$. We also consider the case of antisymmetric forms. Throughout this section, let k be a field of characteristic not equal to two. The notation in this section may be slightly nonstandard.

Most of the definitions and results are taken from [Bak81] and [Sch85, Chapter 7]. Bak works in a much more general setting and defines Hermitian K-theory for non-commutative rings. In Bak's notation, the rings we work with are fields $A = k$, with a trivial involution $a \mapsto \bar{a} = a$, $\lambda\bar{\lambda} = 1$ and $\Lambda = \{0\}$. Since our involutions are trivial, we only have two cases for λ, ± 1.

We recall some basic facts of finite dimensional vector spaces:

- There is a contravariant duality functor

$$^* : \mathrm{Vect}(k) \to \mathrm{Vect}(k),\ V \mapsto \mathrm{Hom}_k(V, k),\ (f : V \to W) \mapsto (g \mapsto gf).$$

- There is a canonical natural transformation $\mathrm{id} \to {}^{**}$.

- The tensor product is left adjoint to the duality functor *.

Definition 2.2.1. A sesquilinear form on a vector space V is a morphism

$$B : V \to \mathrm{Hom}_k(V, k).$$

If B is injective we say that it is a non-singular sesquilinear form (other common notations are non-degenerate and regular). Under the adjunction $\mathrm{Hom}_k(V \otimes V, k) \leftrightarrow \mathrm{Hom}_k(V, \mathrm{Hom}_k(V, k))$ this corresponds to a map $V \otimes V \to \mathrm{Hom}_k(V, k)$. We write B both for the original sesquilinear form $B : V \to V^*$, and its image under this adjunction. That is, $B(x, y) = B(x)(y)$.

Definition 2.2.2. A λ-hermitian form is a non-singular sesquilinear form B such that $B = \lambda B^*$ or $B(x, y) = \lambda B(y, x)$.

With $\lambda = \pm 1$ we recover the usual notion of (anti-)symmetric forms. For symmetric forms we have a simple description. From the polarization identity we can recover a symmetric form from its associated quadratic form, $q_B : V \to k, x \mapsto B(x, x)$, since $\frac{1}{2} \in k$. Note that it is always possible to choose a basis for V such that the matrix representing $B : V \otimes V \to k$ is diagonal (this is constructed by induction, picking a $v \in V\backslash\{0\}$ using $V = \mathrm{Span}\, v \oplus (\mathrm{Span}\, v)^\perp$, [Lam05, Corollary 2.4]). We let $\langle a_1, \ldots, a_n\rangle$ denote the symmetric form with the diagonal $a_1, \ldots, a_n$. This defines the quadratic form $(x_1, \ldots, x_n) \mapsto a_1x_1^2 + \cdots + a_nx_n^2$.

We define the category of λ-hermitian forms $\mathbb{P}^\lambda(k)$, whose objects are pairs of vector spaces and λ-hermitian forms $(V, B : V \to V^*)$. The morphisms are maps of vector spaces which preserve the λ-hermitian forms. That is, for $f : (V, B) \to (V', B')$ we have $f^*B'f = B$, or equivalently $B(x, y) = B'(f(x), f(y))$. Note that since the Hermitian forms are non-singular all the morphism are necessarily injective.

There is a functor $H^\lambda : i\mathbb{P}(k) \to \mathbb{P}^\lambda(k)$ called the hyperbolization functor (here $i\mathbb{P}(k)$ is the subcategory of $\mathbb{P}(k)$ with only isomorphisms as morphisms). It maps a vector space V to $(V \oplus V^*, B)$, where B is the symmetric bilinear form defined by the matrix

$$\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix},$$

or equivalently $B((x \oplus f), (y \oplus g)) = f(y) + \lambda g(x)$. A morphisms $f : V \to W$ is mapped to $f \oplus (f^*)^{-1} : V \oplus V^* \to W \oplus W^*$. Modules in the image of H^λ are called hyperbolic modules. A hyperbolic module of particular importance is the hyperbolic plane, $\mathbb{H} := H^\lambda(k)$.

The category $\mathbb{P}^\lambda(k)$ is symmetric monoidal, with sum

$$(V, B) \oplus (V', B') = (V \oplus V', B \oplus B').$$

There is also a tensor product which respects the monoidal structure

$$(V, B) \otimes (V', B') = (V \otimes V', B \otimes B').$$

Hence, the isomorphism classes become a semiring and are susceptible to Grothendieck completion.

Definition 2.2.3. The zeroth λ-hermitian K-groups of a field k is

$$K_0^\lambda(k) := K_0(\mathbb{P}^\lambda(k)).$$

The symmetric and anti-symmetric forms are particular cases. We set

$$K_0^h(k) := K_0^1(k),$$
$$K_0^{sp}(k) := K_0^{-1}(k).$$

We call $K_0^h(k)$ the zeroth hermitian K-theory ring of k.

The hyperbolization functor induces a natural transformation $K_0 \to K_0^\lambda$. The ring $K_0^h(k)$ is more well known as the Grothendieck-Witt ring $GW(k) := K_0^h(k)$, [Wei13, II.5]. If we take the quotient of this ring by the ideal generated by $\mathbb{H}$ we obtain the Witt ring $W(k) := GW(k)/(\mathbb{H})$. These rings play an important role in number theory.

Example 2.2.4. The Grothendieck-Witt ring can be described in terms of generators and relations [Lam05, Theorem II.4.1]. The Grothendieck-Witt ring is isomorphic to the commutative ring A on generators $\langle x \rangle, x \in k^\times$, with the relations

$$\langle 1 \rangle = 1,$$
$$\langle x \rangle \langle y \rangle = \langle xy \rangle,$$
$$\langle x \rangle + \langle y \rangle = \langle x + y \rangle (1 + \langle xy \rangle), \; x \neq -y.$$

The isomorphism is given by $\langle a_1, \dots, a_n \rangle \mapsto \langle a_1 \rangle + \dots + \langle a_n \rangle$.

For finite fields the Witt ring is ([Lam05, Corollary II.3.6])

$$GW(\mathbb{F}_q) = \begin{cases} \mathbb{Z}/2[\mu_2] & q \equiv 1 \ (4), \\ \mathbb{Z}/4 & q \equiv 3 \ (4). \end{cases}$$

Recall the definition of K_1 as the quotient of $GL(k)$ by the subgroup of elementary matrices $E(k)$. There is a similar definition for λ-hermitian forms. Following [Bak81] we consider the group of automorphisms $\mathrm{Hom}_{\mathbb{P}^\lambda(k)}(H^\lambda(k^n), H^\lambda(k^n))$ as a subgroup of $GL_{2n}(k)$. We define the λ-hermitian general linear groups, $GL_{2n}^\lambda(k) := \mathrm{Hom}_{\mathbb{P}^\lambda(k)}(H^\lambda(k^n), H^\lambda(k^n))$ and let the colimit be $GL^\lambda(k) := \mathrm{colim}_n \, GL_{2n}^\lambda(k)$. Here the colimit is taken as a subgroup of $GL(k) = \mathrm{colim}_n \, GL_n(k)$.

The subgroup of "λ-hermitian-elementary matrices", $E^{\lambda}_{2n}(k)$, is generated by matrices on the form

$$\begin{pmatrix} E & 0 \\ 0 & E^{-1} \end{pmatrix}, \quad \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} I & 0 \\ B & I \end{pmatrix},$$

for E an elementary matrix in $E_n(k)$ and B a λ-hermitian form. With $E^{\lambda}(k) = \operatorname{colim} E^{\lambda}_{2n}(k)$ we set $K^{\lambda}_1 := GL^{\lambda}(k)/E^h(k)$. Note that this is an abelian group, since $E^{\lambda}(k) = [GL^{\lambda}(k), GL^{\lambda}(k)]$ ([Bak81, Corollary 3.9]). The hyperbolization functor induces a natural transformation

$$K_1 \to K^{\lambda}_1.$$

To define K_2 it is possible to define a "λ-hermitian-Steinberg group", $St^{\lambda}(k)$. This is done in [Bak81, Lemma 3.16], together with a map $\mathrm{St}^{\lambda}(k) \to E^{\lambda}(k)$. The definition is a variant of the usual Steinberg group, and is just as enlightening, so we do not repeat the definition here. However, with this in place we set $K^{\lambda}_2(A) := \ker(St^{\lambda}(k) \to E^{\lambda}(k))$.

With the λ-hermitian K^{λ}_1 and K^{λ}_2 groups set up, we consider the particular cases when $\lambda = \pm 1$. Hence, the Hermitian $K^h_i(k)$-groups of k are defined as

$$K^h_i(k) := K^1_i(k), \ i = 0, 1, 2.$$

The corresponding K-groups of anti-symmetric forms are

$$K^{sp}_i(k) := K^{-1}_i(k), \ i = 0, 1, 2.$$

2.3 Higher K-theory

We have only considered algebraic and Hermitian K-theory of fields where 2 is invertible, and only the Hermitian K-groups for $i = 0, 1$ and 2. Later we are going to compute the higher Hermitian K-groups. For completeness we outline the definition of higher algebraic and Hermitian K-theory for smooth schemes over X. We follow [Wei13, IV] and [Hor05, Section 1].

The necessary definitions are:

- Let $\mathcal{C}$ be a small category. The nerve of $\mathcal{C}$ is a simplicial set denote by $N\mathcal{C}$. It is defined by $N\mathcal{C}_n := \{c_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{n-1}} c_n | c_i \in \mathcal{C}\}$, that is the collection of functors $c : [n+1] \to \mathcal{C}$, face maps
$$\delta_i(c_0 \to \cdots \to c_n) := c_0 \to \cdots \to c_{i-1} \xrightarrow{f_n f_{n-1}} c_{i+1} \to \cdots c_n,$$
and coface maps
$$\sigma_i(c_0 \to \cdots \to c_n) := c_0 \to \cdots c_i \xrightarrow{\mathrm{id}} c_i \to \cdots \to c_n.$$

- The bar construction is defined by geometric realization of simplicial sets, $B\mathcal{C} := |N\mathcal{C}|$, [Wei94, 8.1.6].

- For a category $\mathcal{C}$ let $i\mathcal{C}$ be the subcategory with only the isomorphisms as morphisms.

- If $\mathcal{C}$ is a symmetric monoidal category and $\mathcal{C} = i\mathcal{C}$, then we define a category $\mathcal{C}^{-1}\mathcal{C} =: \mathcal{C}^+$, and a functor $\mathcal{C} \to \mathcal{C}^{-1}\mathcal{C}$, such that $B\mathcal{C} \to B\mathcal{C}^{-1}\mathcal{C}$ is a $+$-construction [Wei13, IV.4, IV.1.1].

- For a scheme X, consider the category $\mathrm{Vect}(X)$ of locally free $\mathcal{O}_X$ sheaves of finite rank [Har77, II.5]. It is a symmetric monoidal category with respect to direct sum, and has a duality functor $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$, which we denote by *.

- We define the hermitian category of $\mathrm{Vect}(X)$ as the category with objects pairs
$$(M, \phi : M \xrightarrow{\cong} M^*),$$

with $M \in \text{Vect}(X)$ and ϕ an isomorphism. The morphisms $\alpha : (M, \phi) \to (N, \psi)$ are maps $\alpha : M \to N$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\alpha"] & M^* \\ N \arrow[r, "\psi"] & N^* \arrow[u, "\alpha^*"'] \end{tikzcd}$$

We denote this category by $\text{Vect}(X)_h$.

Definition 2.3.1. Define the algebraic K-theory space, respectively hermitian K-theory space, of X to be

$$K(X) := B(i\text{Vect}(X)^+),$$
$$K^h(X) := B(i\text{Vect}(X)_h^+).$$

The associated homotopy groups are the algebraic K-theory, respectively hermitian K-theory, of X

$$K_n(X) := \pi_n(K(X)),$$
$$K_n^h(X) := \pi_n(K^h(X)).$$

These definitions agree with the definitions of hermitian K-theory given above. The hyperbolic functor induces a continuous map $K(X) \to K^h(X)$.

3 Spectral Sequences

Throughout this chapter all ungraded objects live in some abelian category $\mathcal{A}$, while graded and bigraded objects are defined in the usual way (cf. Appendix A).

3.1 Basics

In this section we define spectral sequences, explain convergence, and discuss spectral sequences of algebra.

Definition 3.1.1. A cohomological spectral sequence is a collection of bigraded objects $\{E_r\}_{r\geq 1}$ called pages, and differentials $d_r : E_r \to E_r, d_r \circ d_r = 0$ of bidegree $(r, -r+1)$, such that $E_{r+1} \cong H(E_r, d_r)$. That is

$$E_{r+1}^{s,t} \cong \ker d_r^{s,t} / \operatorname{im} d^{s+r,t-r-+1}.$$

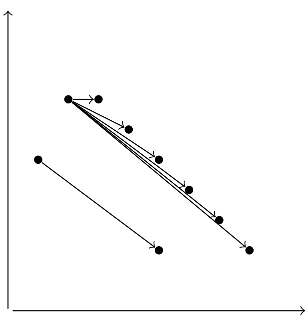
A map of spectral sequences is a collection of maps $f_r : E_r \to E'_r$ which are compatible with the differentials,

$$\begin{array}{ccc} E_r^{s,t} & \xrightarrow{d_r^{s,t}} & E_r^{s+r,t-r+1} \\ \downarrow{\scriptstyle f_r^{s,t}} & & \downarrow{\scriptstyle f_r^{s+r,t-r+1}} \\ {E'}_r^{s,t} & \xrightarrow{{d'}_r^{s,t}} & {E'}_r^{s+r,t-r+1} \end{array}$$

The map f_{r+1} must be induced by f_r. Note that if f_r is an isomorphism for some $r = r_0$, then f_r is an isomorphism for all $r \geq r_0$.

Remark 3.1.2. There are also homologically graded spectral sequences defined dually to the above definition. We leave this out as all our spectral sequences are cohomologically graded.

To keep track of all the data in a spectral sequence it is common to draw a page in a two dimensional plane. The group $E_r^{s,t}$ is drawn at coordinates (s,t). The d_r-differential goes one square to the left and the r steps downwards along a line of slope -1. This is indicated in the figure below. Often other conventions are used to draw the images. That is, the image is subject to some transformation in $\mathrm{GL}(\mathbb{Z})$.



All the subsequent pages in a spectral sequence can be identified as quotients of submodules the E_1-page [MP12, 24.5]. Under this identification we let B_r be the d_r-boundaries and Z_r the d_r-cycles, and we get a sequence of submodules

$$0 = B_0 \subset B_1 \subset \ldots \subset Z_2 \subset Z_1 \subset Z_0 = E_1.$$

Define $Z_\infty := \cap Z_r$ and $B_\infty := \cup B_r$. Then the E_∞-page is defined as

$$E_\infty := Z_\infty / B_\infty.$$

If in each bidegree (s,t) there exists some r_0 such that $d_r^{s,t} = 0$ and $d_r^{p-r,q+r-1} = 0$ for all $r \geq r_0$, the spectral sequence is said to have collapsed in this bidegree. We have $E_r^{s,t} \cong E_{r_0}^{s,t}$ for all $r \geq r_0$, and define $E_\infty^{s,t} := E_{r_0}^{s,t}$. Most of our spectral sequences are such that in each bidegree they collapse after a finite number of steps.

Definition 3.1.3 ([Boa99, Definition 5.2]). A cohomological spectral sequence is said to converge to a graded object A called the abutment, if there is a decreasing filtration $F^{s+1} \subset F^s A$ on A such that

1. The filtration is exhaustive, $\cup_s F^s A = A$.

2. The filtration is Hausdorff, $\cap_s F^s A = 0$.

3. There are isomorphisms $F^s A^{s+t}/F^{s+1}A^{s+t} \cong E_\infty^{s,t}$.

To indicate convergence it is commmon to write $E_r \implies A$. A spectral sequence is said to converge strongly if the filtration of A is complete. That is, the induced topology on A is complete.

Remark 3.1.4. Note that in the pictorial representation of a spectral sequence given above, the quotients in the filtration of A^{s+t} all lie along a diagonal $s+t=C$. If each diagonal only contain finitely many nonzero terms, the filtration must be finite in that degree. If this holds for all diagonals, and the filtration is exhaustive and Hausdorff, we automatically have strong convergence to the abutment.

If we have some spectral sequence E_r converging strongly to some graded object A, and have computed the E_∞-page $\implies A$, this is in general not enough to determine A. This is because we only know the quotients $F^{s+1}A/F^s A$. We have to be able to solve uniquely all the extension problems

$$0 \to F^{s+1}A \to F^s A \to F^s A/F^{s+1}A \to 0.$$

To manage this we consider spectral sequences with more structure, spectral sequences of algebras. A spectral sequence of algebras is a spectral sequence E_r with a pairing on each page $E_r \otimes E_r \to E_r$, such that the pairing on the E_{r+1}-page is induced by the pairing on the E_r-page, and we have a Leibniz rule:

$$d_r(xy) = d_r(x)y + (-1)^p x d_r(y), \quad x \in E_r^{p,q}.$$

We say that the spectral sequence converges to an algebra A if the spectral sequence converge to A, and if the pairing induced on the E_∞-page is compatible with the pairing induced by the filtration. That is, the following diagram is commutative

$$\begin{tikzcd}
(F^s A^{s+t}/F^{s+1}A^{s+t}) \otimes (F^p A^{p+q}/F^{p+1}A^{p+q}) \arrow[r] \arrow[d, "\cong"] & F^{p+s}A^{s+t+p+q}/F^{p+s+1}A^{s+t+p+q} \arrow[d, "\cong"] \\
E_\infty^{s,t} \otimes E_\infty^{p,q} \arrow[r] & E_\infty^{p+s,q+t}
\end{tikzcd} \tag{3.1}$$

With this extra structure it becomes possible to solve several extension problems.

The construction of a spectral sequence is usually functorial. In such cases the theorem below can be used to deduce that a map is an isomorphisms on the abutments if it induces an isomorphism on the E_∞-pages.

Theorem 3.1.5 ([Boa99, Theorem 2.6]). *Let $f : G \to \bar{G}$ be a morphism of filtered groups such that:*

1. *The filtrations are exhaustive.*

2. *f induces $F^\infty G \cong F^\infty \bar{G}$.*

3. The filtration on G is complete.

4. f induces isomorphisms $F^sG/F^{s+1}G \cong F^s\bar{G}/F^{s+1}\bar{G}$.

Then f is an isomorphism of filtered groups.

Example 3.1.6. The Serre spectral sequence is a classic of topology. Given a fiber sequence $F \to E \to B$, where B is simply connected, it is a cohomological spectral sequence of algebras such that

$$E_2^{p,q} = H^p(B, H^q(F)) \implies H^{p+q}(E).$$

Using this we can for instance calculate the cohomology of the complex projective n-space $\mathbb{CP}^n$. We have a fiber sequence $S(1) = U(1) \to S^{2n+1} \to \mathbb{CP}^n$. The Künneth theorem tells us that the E_2-page is $H^*(B) \otimes H^*(F)$, compatible with the algebra structure. Therefore the E_2-page is concentrated along the rows 0 and 1, and only the d_2-differential can be nonzero. Hence, the E_3-page is the E_∞-page. The E_2-page of the Serre spectral sequence of $\mathbb{CP}^4$ is drawn in Example 3.1.6. The abutment is H^*S, which is $\mathbb{Z}$ in degree 0 and $2n+1$, and 0 otherwise. This implies that most of the E_∞-page is 0, so d_2 must be an isomorphism in all degrees except when the target is $E_2^{0,0}$ and $E_2^{2n+1,0}$. In addition, because of the Künneth theorem, $E_2^{i,0} \cong E_2^{i,1}$. From Example 3.1.6 we immediately see that $H^{2k+1}\mathbb{CP}^n = 0$ and $H^{2k}\mathbb{CP}^n = \mathbb{Z}, 0 \leq k \leq n$, since $H^0\mathbb{CP} = \mathbb{Z}$, and this stops in degree $2n$, for if $d_2^{2n,1}$ was nonzero, then the filtration of $H^{2n+2}S$ would be nonzero. With this knowledege it is easy to compute the cup-product structure of $\mathbb{CP}^n$. Let $l \in E_2^{0,1}$ and $x \in E_2^{2,0}$ be generators such that $d_2(l) = x$. Using the Leibniz rule we get by induction that $d_2(lx^k) = d_2(l)x^k = x^{k+1}$ is a generator of $H^{2(k-1)}\mathbb{CP}^n$. This is the cup-product structure of $\mathbb{CP}^n$.

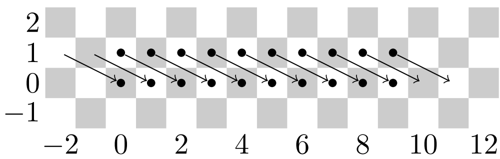


Figure 3.1: The E_2-page of the Serre spectral sequence for $\mathbb{CP}^4$. Each dot is a generator.

3.2 Exact Couples

In this section we briefly discuss exact couples. They are our main tool for constructing spectral sequences.

Definition 3.2.1. An exact couple is a pair of bigraded objects (D, E) in $\mathcal{A}$ with maps

$$D \xrightarrow{i} D,\ D \xrightarrow{j} E,\ E \xrightarrow{k} E,$$

such that $\ker i = \operatorname{im} k, \ker j = \operatorname{im} i, \ker k = \operatorname{im} j$. It is common to display an exact couple as

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

A map of exact couples $(D, E) \to (D', E')$ is a triple of maps (f, g, h) such that the diagram commutes:

$$\begin{tikzcd} D \arrow[rr, "i"] \arrow[rrrr, bend left, "f"] & & D \arrow[dl, "j"] \arrow[rrrr, bend left, "g"] & & D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] \arrow[rrrr, "h"] & & & & E' \arrow[ul, "k"] & \end{tikzcd}$$

When we have an exact couple (D, E), we may take the homology of E with respect to the map jk, since $jkjk = 0$. This gives rise to a new exact couple $C_1 := (i(D), H(E, jk))$ with maps induced from the old exact couple (D, E). Proceeding like this we get a sequence of exact couples $(C_i)_{i \geq 0}$, where $C_0 = (D, E)$, and the couple C_{i+1} is obtained from C_i by the procedure above. If the maps are graded, we can combine all the information to form a spectral sequence. We restrict to the case when our abelian category is graded, and the morphisms i and j have degree 0, while k has degree 1, similar to [Boa99]. In this case we can "unroll" the exact couple and obtain a diagram called an unrolled exact couple

$$\begin{tikzcd}
\cdots \arrow[r, "i"] & A^{i+1} \arrow[rr, "i"] & & A^{i} \arrow[rr, "i"] \arrow[dl, "j"] & & A^{i-1} \arrow[r, "i"] \arrow[dl, "j"] & \cdots \\
& & E_i \arrow[ul, "k"] & & E_0 \arrow[ul, "k"] & &
\end{tikzcd}$$

Definition 3.2.2 ([Boa99])**.** There is a spectral sequence constructed as follows. Define

$$\begin{aligned}
Z_r^s &= k^{-1}(\mathrm{im}[i^{(r-1)} : A^{s+r} \to A^{s+1}]), \\
B_r^s &= j \ker i^{(r-1)} : A^s \to A^{s-r+1}, \\
E_r^s &= Z_r^s / B_r^s.
\end{aligned}$$

The differentials are given by

$$d_r : E_r^s \twoheadrightarrow Z_r^s / \ker k \hookrightarrow \mathrm{im}\left[i^{(r-1)} : A^{s+r} \to A^{s-1}\right] \twoheadrightarrow \mathrm{im}\, j / B_r^{s+r} \hookrightarrow E_r^{s+r}$$

where the last surjective map is given by lifting $i^{(r-1)}$ and applying j. Note that the d_1 differential is equal to jk.

It is now possible to filter $A^{-\infty} := \mathrm{colim}_{s \to -\infty} A^s$ by the image filtration

$$F^s A := \mathrm{im}[A^s \to A^{-\infty}].$$

Boardman discusses several criterions for the convergence of the above spectral sequence above. We restrict to the special case of the following theorem, which most of our spectral sequences satisfy.

Theorem 3.2.3. *Let (A, E) be an unrolled exact couple as above. Suppose that*

- *The maps $A^0 \xrightarrow{i} A^i, i < 0$ are the identity (i.e., the exact couple is cut off). Hence, $A^{-\infty} = A^0$.*
- *The groups $E_r^{s,t}$ are finitely generated.*
- *The limits $A^{\infty} := \lim_s A^s$ and $\lim_s^1 A^{\infty}$ are zero (i.e., the spectral sequence is conditionally convergent* [Boa99, Definition 5.10]*).*

Then the associated spectral sequence is strongly convergent.

Proof. This is essentially [Boa99, Theorem 7.3]. □

3.3 The Bockstein Spectral Sequence

Let (A, d) be a differential graded algebra (Definition A.5.1) with a decreasing filtration $F^s A$, which is Hausdorff, complete and compatible with the differential and algebra structure, and $F^0 A = A$.

With this data we can construct an unrolled exact couple as follows. Consider the short exact sequence

$$0 \to F^{s+1} A \xrightarrow{i} F^s A \xrightarrow{j} F^s A / F^{s+1} A \to 0.$$

Taking homology we get an unrolled exact couple

$$\begin{tikzcd}
\cdots \arrow[r] & H(F^{s+2}A) \arrow[rr, "i"] & & H(F^{s+1}A) \arrow[rr, "i"] \arrow[dl, "j"] & & H(F^sA) \arrow[r] \arrow[dl, "j"] & \cdots \\
& & H(F^{s+1}A/F^{s+2}A) \arrow[ul, "\delta"] & & H(F^sA/F^{s+1}A) \arrow[ul, "\delta"] & &
\end{tikzcd}$$

By Definition 3.2.2 and Theorem 3.2.3, we have a spectral sequence converging strongly to $H(F^0A) = H(A)$.

Theorem 3.3.1. *Let (A, d) be a differential graded algebra with a decreasing filtration F^sA as above. Then there is a strongly convergent spectral sequence*

$$E_1^{s,t} = H^{s+t}(F^sA/F^{s+1}A) \implies H^{s+t}(A).$$

The d_1 differential is induced by the differential d of A. The algebra structure on the E_1-page is induced by the pairing

$$F^sA/F^{s+1}A \otimes F^tA/F^{t+1}A \to F^{s+t}A/F^{s+t+1}A.$$

Of particular importance for us is the filtration obtained from a central element $x \in A$, such that $d(x) = 0$ (i.e., the differential is x-linear). This gives a decreasing filtration $F^sA := x^sA$. This is Hausdorff if and only if no element is infinitely x-divisible. The filtration is complete if x is of positive degree, and A is zero in negative degrees. If we apply Theorem 3.3.1, the associated spectral sequence is called the x-Bockstein spectral sequence of A.

Example 3.3.2. . The x-Bockstein spectral gets its name because it is constructed in the same manner as the ordinary Bockstein spectral sequence (and also a lot of other Bockstein spectral sequences, e.g., [Wei94, 5.9.9] or [McC01, Chapter 10]). This is the spectral sequence obtained from the exact sequence of coefficients

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0.$$

Taking homology with respect to some space X in these coefficients we get an exact couple

$$\begin{tikzcd}
H_*(X) \arrow[rr, "p"] & & H_*(X) \arrow[dl, "j"] \\
& H_*(X;\mathbb{Z}/p) \arrow[ul, "\delta"] &
\end{tikzcd}$$

If X is a connected space of finite type, this is a spectral sequence with $E_1 = H_*(X;\mathbb{F}_p)$, converging to $(H_*(X)/\text{torsion}) \otimes \mathbb{F}_p$. The d_1 diffferential is by definition β, the Bockstein homomorphism.

3.4 The Motivic Adams Spectral Sequence

In this section we construct the motivic Adams spectral sequence, identify the E_2-page and discuss convergence. It is essentially the same construction as in topology, see for instance [Rav86, Chapter 2]. This construction has been generalized to the motivic world by several others, see for example [Mor99] or [DI10]. The results of this section are summarized in Proposition 3.4.1.

The construction of the motivic Adams spectral sequence may be carried out over any ring spectrum [Rav86, 2.2], and X any spectrum. We do not pursue this more general approach, since we are only after the results for the Eilengberg-Maclane spectrum H.

Proposition 3.4.1. *The motivic Adams spectral sequence is a trigraded spectral sequence with E_2-page*

$$\mathrm{Ext}_{A_\star}(H_\star, H_\star X)$$

and differential

$$d_r : E_r^{s,m+n\alpha} \to E_r^{s+r,m+r-1+n\alpha}.$$

If X is a cellular spectrum of finite type (Definition 1.6.4) the spectral sequence converges strongly to $\pi_\star X_{\hat{2}}$. The convergence is given by $E_\infty^{s,m+n\alpha} \implies \pi_{m-s+n\alpha} X_{\hat{2}}$. On the E_2-page s is the homological degree of Ext.

Remark 3.4.2. The Adams spectral sequence is of "Adams-type". This is a cohomological spectral sequence if we flip the sign of the second index.

We now sketch the construction and the identification of the E_2-page.

Definition 3.4.3. An Adams resolution for a spectrum X is a sequence of spectra X_i and maps

$$\begin{tikzcd} \cdots \arrow[r] & X_k \arrow[r, "g_{k-1}"] \arrow[d, "f_k"] & X_{k-1} \arrow[r, "g_{k-2}"] \arrow[d, "f_{k-1}"] & \cdots \arrow[r] & X_1 \arrow[r, "g_0"] \arrow[d, "f_1"] & X_0 = X \arrow[d, "f_0"] \\ & K_k & K_{k-1} & & K_1 & K_0 \end{tikzcd} \tag{3.2}$$

Such that the following holds

1. Each $X_{s+1} \xrightarrow{g_s} X_s \xrightarrow{f_s} K_s$ is a cofiber sequence.
2. $H \wedge X_s \xrightarrow{H\wedge f_s} H \wedge K_s$ has a retraction. This implies that $H_\star f_s$ is a monomorphism.
3. $K_s = S \wedge K_s \xrightarrow{\eta\wedge K_s} H \wedge K_s$ has a retraction.
4. There is a natural isomorphism

$$\mathrm{Ext}_{A_\star}^{t,\star}(H_\star, H_\star K_s) = \begin{cases} \pi_\star K_s & t = 0, \\ 0 & t \neq 0. \end{cases}$$

   where the isomorphism in homological degree 0 is given by the natural map

$$\pi_\star K_s \ni f \mapsto \pi_\star H \wedge f \in \mathrm{Hom}_{A_\star}(H_\star, H_\star K_s).$$

Definition 3.4.4 ([Rav86, 2.2.9])**.** The canonical Adams resolution is defined inductively by

$$\begin{aligned} X_0 &:= X, \\ K_s &:= H \wedge X_s, \\ X_s &= S \wedge X_s \xrightarrow{\eta\wedge X_s} H \wedge X_s, \\ X_{s+1} &:= \mathrm{hofib}(X_s \xrightarrow{f_s} K_s). \end{aligned}$$

Lemma 3.4.5. *The canonical Adams resolution is an Adams resolution.*

Proof. We check that all the properties in Definition 3.4.3 are satisfied. Property (1) is satisfied by construction. The map $H \wedge X_s \xrightarrow{H\wedge f_s} H \wedge H \wedge X_s$ has a retraction via the map $\mu \wedge X_s$, so we have property (2). Similarly property (3) is via $H \wedge K_s \xrightarrow{\epsilon\wedge K_s} K_s$. By Lemma 1.3.3

$$H_\star H \otimes_{H_\star} H_\star X_s \xrightarrow{\cong} H_\star H \wedge X_s,$$

that is, $H_\star H \wedge X_s$ is a relative injective $H_\star H$-comodule. Together with Lemma A.2.5 we then get a series of isomorphisms

$$\mathrm{Ext}_{A_\star}(H_\star, H_\star K_s) \cong \mathrm{Ext}_{A_\star}(H_\star, H_\star H \otimes_{H_\star} H_\star X_s) \cong \mathrm{Hom}_{H_\star}(H_\star, H_\star X_s) \cong H_\star X_s = \pi_\star X_s,$$

concentrated in homological degree 0. Hence, property (4) is satisfied. $\square$

Taking the long exact sequence of the resolution in Equation (3.2) yields the unrolled exact couple

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_\star X_k \arrow[r, "\pi_\star g_{k-1}"] \arrow[d, "\pi_\star f_k"] & \pi_\star X_{k-1} \arrow[r, "\pi_\star g_{k-2}"] \arrow[d, "\pi_\star f_{k-1}"] & \cdots \arrow[r] & \pi_\star X_1 \arrow[r, "\pi_\star g_0"] \arrow[d, "\pi_\star f_1"] & \pi_\star X_0 = \pi_\star X \arrow[d, "\pi_\star f_0"] \\
& \pi_\star K_k \arrow[ul, "\partial"] & \pi_\star K_{k-1} \arrow[ul, "\partial"] & & \pi_\star K_1 \arrow[ul, "\partial"] & \pi_\star K_0 \arrow[ul, "\partial"]
\end{tikzcd}$$

By Definition 3.2.2 we get a spectral sequence. The E_1-page and the differential are as always $E_1 = \pi_\star K_k$ and $d_1 = \pi_\star f_k \partial$.

The usefulness of the Adams spectral sequence is partly because there is a simple identification of the E_2-page as

$$E_2 = \mathrm{Ext}_{A_\star}(H_\star, H_\star X). \tag{3.3}$$

If the structure of $H_\star$ and $H_\star X$ as $A_\star$-comodules are known, the E_2-page can in principle always can be computed. We arrive at this by considering the long exact sequence in homology of the fiber sequences in an Adams resolution. Since each $H_\star(f_s)$ is assumed to be a monomorphism they split into short exact sequences

$$0 \longrightarrow H_\star X_k \xrightarrow{H_\star f_k} H_\star K_k \xrightarrow{\partial} H_{\star-1} X_{k+1} \longrightarrow 0$$

that patch together

$$\underline{0} \longrightarrow \underline{H_\star X_0} \rightarrowtail \underline{H_\star K_0} \twoheadrightarrow H_{\star-1} X_1 \rightarrowtail \underline{H_{\star-1} K_1} \twoheadrightarrow H_{\star-2} X_2 \rightarrowtail \underline{H_{\star-2} K_2} \twoheadrightarrow H_{\star-3} X_3 \longrightarrow \cdots$$

The underlined terms form a long exact sequence

$$0 \longrightarrow H_\star X \longrightarrow H_\star K_0 \longrightarrow H_\star \Sigma K_1 \longrightarrow H_\star \Sigma^2 K_2 \longrightarrow H_\star \Sigma^3 K_3 \longrightarrow \cdots$$

By property (4) of Definition 3.4.3 this is a resolution of $H_\star X$ by relatively injective comodules. Applying $\mathrm{Hom}_{A_\star}(H_\star, -)$ to the resolution yield the commutative diagram with vertical isomorphism

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{A_\star}(H_\star, H_\star K_0) \arrow[r] \arrow[d] & \mathrm{Hom}_{A_\star}(H_\star, H_\star \Sigma K_1) \arrow[r] \arrow[d] & \mathrm{Hom}_{A_\star}(H_\star, H_\star(\Sigma^2 K_2)) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \pi_\star K_0 \arrow[r] & \pi_\star \Sigma K_1 \arrow[r] & \pi_\star \Sigma^2 K_2 \arrow[r] & \cdots
\end{tikzcd}$$

The bottom row is the E_1-page of the Adams spectral sequence. Taking homology we obtain the E_2-page. Taking homology of the top row we get $\mathrm{Ext}_{A_\star}(H_\star, H_\star X)$, and we have arrived at the identification of the E_2-page.

Given a map of ring spectra $X \xrightarrow{f} Y$ with Adams resolutions (X_s, K_s) and (Y_s, L_s) there exists a map of exact couples (f_s, g_s) lifting f. This lifting is, in a specific sense, unique up to homotopy. We get induced maps of spectral sequences of algebras, which on the E_2-page is given by $\mathrm{Ext}_A(H_\star, H_\star X) \xrightarrow{\mathrm{Ext}_A(H_\star, H_\star f)} \mathrm{Ext}_A(H_\star, H_\star Y)$. Hence, from the E_2-page and onwards it is independent of the lifting (f_s, g_s), so the motivic Adams spectral sequence is functorial in X, for $E_r, r \geq 2$. An easy corollary of this is:

Corollary 3.4.6. *If $X \simeq Y$, then $E_r X \cong E_r Y, r \geq 2$.*

Hu, Kriz and Ormsby [HKO11] has proved that the motivic Adams spectral sequence is strongly convergent over fields of characteristic 0. This can be generalized to smooth schemes over general fields by the description of the structure of the motivic Steenrod algebra over fields with positive characteristic as carried out in [HKØ13]. This is stated in [HKØ13] after Proposition 5.5.

Proposition 3.4.7 ([HKO11, Corollary 3], [HKØ13, Section 5]). *If X is a motivic cell spectrum of finite type, then the motivic Adams spectral sequence is strongly convergent to $\pi_\star X_{\hat{2}}$.*

If the spectrum X above is a ring spectrum, the Adams spectral sequence becomes a spectral sequence of algebras. This algebra structure is used several places, for instance [Orm11], [Hil11] and [OØ13].

Theorem 3.4.8 ([Rav86, Theorem 2.3.3]). *Let X be a ring spectrum satisfying the assumptions of Proposition 3.4.1. Then the motivic Adams spectral sequence is a spectral sequence of algebras $2 \leq r \leq \infty$, converging to the algebra $\pi_\star X_{\hat{2}}$. For $r = 2$ the pairing is the external cup-product (Appendix A.4.1) composed with the map obtained by applying* $\mathrm{Ext}_{A_\star}(H_\star, -)$ *to*

$$H_\star X \otimes_{H_\star} H_\star X \to H_\star X \wedge X \xrightarrow{H_\star \mu} H_\star X.$$

Example 3.4.9. The motivic Adams spectral sequence is a generalization of the classical Adams spectral sequence in topology. For some connective spectrum X of finite type this is a strongly convergent bigraded spectral sequence with $E_2 = \mathrm{Ext}_{A_*}(H_*, H_*X) \implies \pi_* X_{\hat{2}}$. Here H_*, H_*X and A_* are singular homology and the topological Steenrod algebra. If X is a ring spectrum then it is a spectral sequence of algebras.

The topological analogy of hermitian K-theory is real K-theory. This theory is represented by a spectrum KO. If we set up the Adams spectral sequence for ko, the connective cover of KO, the E_2-page is the algebra

$$E_2 = \mathbb{F}_2[h_0, h_1, v, w_1]/(h_0h_1, h_1^3, vh_1, v^2 - h_0^2 w_1),$$

where the generators have the grading

generator	grading
h_0	$(1,1)$
h_1	$(1,2)$
v	$(3,7)$
w_1	$(4,12)$

The E_2-page is drawn in Figure 3.2. The spectral sequence actually collapses at the E_2-page.

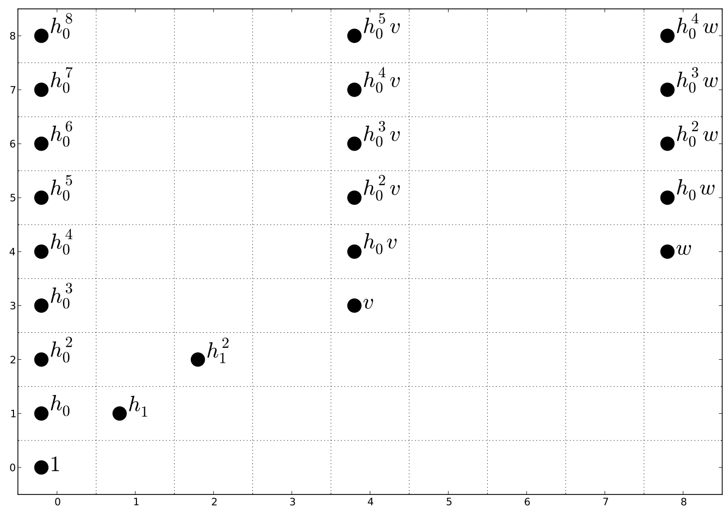


Figure 3.2: An Adams chart for the E_2-page of the topological spectrum KO. The convention in Adams charts is to draw generators in degree (s,t) at the coordinates $(t-s, s)$. Then the d_r-differential goes one step to the left and r steps up.

From the Adams chart (Figure 3.2) it is clear that the only possibility for a differential on a generator is on h_1. If we assume that h_1 support a d_r-differential, then $d_r(h_1) = h_0^r$. Then

$$h_0 h_0^r = h_0 d_r(h_1) = d_r(h_0 h_1) = 0,$$

a contradiction. Hence, we have convergence already at the E_2-page. It is then straightforward to read off the convergence. All the possible extensions are connected by h_0. In $\pi_* ko$, the class h_0 is represented by 2. This implies

$$\pi_*(ko)_{\hat{2}} = \mathbb{Z}_2[h_1, v, w_1]/(2h_1, h_1^3, h_1 v, v^2 - 4w).$$

3.5 The Slice Spectral Sequence

Recall the effective functors f_q defined in Definition 1.5.4. If we take the cofiber of the natural map $f_{q+1}E \to f_q E$, we get a cofiber sequence

$$f_{q+1}E \to f_q E \to s_q E \to \Sigma f_{q+1} E. \tag{3.4}$$

It turns out that this construction is functorial, that is, the s_q are triangulated functors. They are called the slice functors. If we consider the long exact sequence on homotopy groups induced by the cofiber sequences as q vary, we get an unrolled exact couple:

$$\begin{tikzcd} \cdots \arrow[r] & \pi_\star f_{q+1}E \arrow[rr] & & \pi_\star f_q E \arrow[rr] \arrow[dl] & & \pi_\star f_{q-1}E \arrow[r] \arrow[dl] & \cdots \\ & & \pi_\star s_q E \arrow[ul] & & \pi_\star s_{q-1}E \arrow[ul] & & \end{tikzcd}$$

The associated spectral sequence is the slice spectral sequence ([Voe02b, Section 7]). Convergence of the slice spectral sequence is in general an open and hard problem. However, for certain base schemes and spectra we have convergence to $\pi_\star E$. The filtration of $\pi_\star E$ is induced from the canonical maps $f_q E \to E$ and is defined as $f_q \pi_\star E := \mathrm{im}(\pi_\star f_q E \to E) \subset \pi_\star E$.

Proposition 3.5.1 ([Lev13b, Theorem 4]). *Let k be a finite field with characteristic p. Let E be a finite spectrum. Then the slice spectral sequence for E with p inverted is convergent (by inverting p we mean that the homotopy groups taking part in the spectral sequences are all localized such that p is invertible).*

Proof. From [Lev13b, Theorem 4] we know that the spectral sequence is convergent in the sense of [Voe02b, Definition 7.1]. Because of Lemma 1.5.6 and Lemma 1.1.16,

$$\pi_\star E \cong \pi_\star \operatorname{hocolim}_q f_q E = \operatorname{colim}_q \pi_\star f_q E = \operatorname{colim}_q f_q \pi_\star E.$$

Hence, the filtration is exhaustive. □

Much can be said about the convergence of the slice spectral sequence. We will only need that it is strongly convergent for $\mathbf{BP}\langle 1 \rangle$, and convergent for KGL.

Proposition 3.5.2 ([Hoy13, Theorem 8.12], [Voe02a, Section 5]). *The slice spectral sequence for $\mathbf{BP}\langle n \rangle$ is strongly convergent for all n. The slice spectral sequence for KGL is strongly convergent.*

4 The Motivic Homotopy Groups of KO

In this section we compute the motivic homotopy groups of **KO**, that is, $\pi_\star ko_{\hat{2}}$. Since **KO** represents Hermitian K-theory, Equation (1.3) give us the Hermitian K-theory of finite fields. The homotopy groups are computed using the motivic Adams spectral sequence, Proposition 3.4.1. We do the computation in parallel for $H\mathbb{Z}_{(2)}$, ko and kgl. We computed the homotopy groups $\pi_\star H\mathbb{Z}_{\hat{2}}$ in Example 1.2.7. This was a consequence of the computation of the algebraic K-theory of finite fields, due to [Qui72]. This determines the differentials in the motivic Adams spectral sequence for $\pi_\star(H\mathbb{Z}_{(2)})$. The morphisms defined in Section 1.3, between ko, kgl and $H\mathbb{Z}_{(2)}$, then determine the differentials in the Adams spectral sequences for kgl and ko.

The outline of this section is as follows. First we compute the E_2-page of the motivic Adams spectral sequence. This is achieved by comparison with computations over algebraically closed fields and by use of the ρ-Bockstein spectral sequence. Since we know the abutment for the motivic Adams spectral sequence of $H\mathbb{Z}_{(2)}$, we can determine all the differentials in the spectral sequence. Comparing $H\mathbb{Z}_{(2)}$ with kgl and ko, and using the multiplicative structure, we obtain the differentials of kgl and ko. Finally we compute the E_∞-pages for ko and resolve the extension problems. Our results agree with the results of Friedlander, [Fri76].

4.1 The E_2-page

Equation (3.3) states that the E_2-page of the motivic Adams spectral sequence for a spectrum X is

$$E_2 = \mathrm{Ext}_{A_\star}(H_\star, H_\star X).$$

The abutment is $\pi_\star X_{\hat{2}}$.

Remark 4.1.1. For strong convergence we need the spectrum X to be cellular of finite type. Strictly speaking this is not true for any of the spectra ko, kgl or $H\mathbb{Z}_{(2)}$. We would need to replace the notion of S-cellularity with $S_{(2)}$-cellularity, or work in a localization of the motivic stable homotopy category, cf. [Lev13a]. For ko we do not know whether this works. However, we would expect the connected cover of **KO** to have a finite cell structure, with only $(m+n\alpha)$-cells for $m, n \geq 0$. This is true in topology, and under these assumptions our computations are correct. These finiteness conditions on the spectra also imply that the application of Lemma 1.7.10 is correct (cf. [Ada74, 13.2]).

From the computations in Section 1.8 we have

$$\begin{aligned} H_\star\left(\mathbb{Z}_{(2)}\right) &= A_\star \,\square_{E_\star(0)}\, H_\star, \\ H_\star(kgl) &= A_\star \,\square_{E_\star(1)}\, H_\star, \\ H_\star(ko) &= A_\star \,\square_{A_\star(1)}\, H_\star. \end{aligned}$$

Application of Proposition A.3.2 with $(A, \Gamma) = (H_\star, A_\star)$ and $(B, \Sigma) = (H_\star, B)$ then yields

$$\mathrm{Ext}_{A_\star}(H_\star, A_\star \,\square_B\, H_\star) \cong \mathrm{Ext}_B(H_\star, H_\star)$$

for $B = E_\star(0), E_\star(1)$ and $A_\star(1)$. We now have two cases depending on the class of q modulo 4, that is, wether ρ is trivial or not.

4.1.1 $q \equiv 1$ (4)

Let $(\overline{H}_\star, \overline{B}_\star)$ denote one of the Hopf algebroids above, defined over an algebraically closed field. Recall from Proposition 1.2.5 that $\overline{H}_\star = \mathbb{Z}/2[\tau]$ and $\rho = 0$. When $q \equiv 1$ (4) we have that

$\rho = [-1] \in k^\times/(k^\times)^2$ is trivial. Then the algebraic closure map $\mathbb{F}_q \to \overline{\mathbb{F}}_q$ induces a map of Hopf algebroids $(\overline{H}_\star, \overline{B}_\star) \to (H_\star, B_\star)$ such that $H_\star = \overline{H}_\star\{1, u\}$ and $\overline{B}_\star \otimes_{\overline{H}_\star} H_\star \cong B_\star$. Hence, we may apply Corollary A.3.3 to obtain

$$\mathrm{Ext}_B(H_\star, H_\star) \cong H_\star \otimes_{\overline{H}_\star} \mathrm{Ext}_{\overline{B}}(\overline{H}_\star, \overline{H}_\star). \tag{4.1}$$

Theorem 4.1.2. *Over an algebraically closed field we have*

$$\begin{aligned}
\mathrm{Ext}_{\overline{E}_\star(0)}(\overline{H}_\star, \overline{H}_\star) &= \overline{H}_\star[h_0],\\
\mathrm{Ext}_{\overline{E}_\star(1)}(\overline{H}_\star, \overline{H}_\star) &= \overline{H}_\star[h_0, v_1],\\
\mathrm{Ext}_{\overline{A}_\star(1)}(\overline{H}_\star, \overline{H}_\star) &= \frac{\overline{H}_\star[h_0, h_1, v, w_1]}{(h_0h_1, \tau h_1^3, h_1 v, v^2 - h_1^2 w_1)}.
\end{aligned}$$

The bidegrees of the generators are:

generator	bidegree
τ	$(0, 1-\alpha)$
h_0	$(1, 1)$
v_1	$(1, 2+\alpha)$
h_1	$(1, 1+\alpha)$
v	$(3, 5+2\alpha)$
w_1	$(4, 8+4\alpha)$

Proof. We have $\overline{E}_\star(0) = \mathbb{Z}/2[\tau, \tau_0]/(\tau_0^2)$. This is almost an exterior algebra, for which the computation of Ext is well known. The same techniques are almost directly applicable in this case too. We use the cobar complex of the pair $(\overline{H}_\star, \overline{H}_\star)$ over $\overline{E}_\star(0)$ (Definition A.4.1), which in this case is very simple. In the notation of Definition A.4.1, let $\Gamma = \overline{E}_\star(0)$ and $A = \overline{H}_\star = \mathbb{Z}/2[\tau]$. Then $\overline{\Gamma} = A\{\tau_0\}$ and the n-th term of the cobar complex is $A\{1 \otimes \tau_0^{\otimes n} \otimes 1\}$. We claim that all the differentials in the cobar complex are 0. This is easily computed, since in this case the differentials are A-linear and preserve the bidegree. But the bidegrees of the generators are strictly increasing (i.e., $|1 \otimes \tau_0^n \otimes 1| < |1 \otimes \tau_0^{n+1} \otimes 1|$), hence, the differentials are zero (alternatively one can use that $\Delta\tau_0 = \tau_0 \otimes 1 + 1 \otimes \tau_0$ and compute the differentials directly). From the product structure described in Appendix A.4.1, it is clear that $\mathrm{Ext}_{\overline{E}_\star(0)}(\overline{H}_\star, \overline{H}_\star) = \overline{H}_\star[h_0]$, where $h_0 = \tau_0$ in the cobar complex.

Observe that $\overline{E}_\star(1) = \mathbb{Z}/2[\tau, \tau_0, \tau_1]/(\tau_0^2, \tau_1^2)$. Hence, we have $\overline{E}_\star(1) = \overline{E}_\star(0)\{1, \tau_1\}$ as an $\overline{E}_\star(0)$-module. Consider complexes $P_\bullet$ and $Q_\bullet$ with homology

$$\mathrm{Ext}_{\overline{E}_\star(0)}(\overline{H}_\star, \overline{H}_\star) \quad \text{and} \quad \mathrm{Ext}_{\overline{E}_\star(0)\{\tau_1\}}(\overline{H}_\star, \overline{H}_\star) \quad \text{respectively.}$$

Then $P_\bullet \otimes Q_\bullet$ is a complex with homology $\mathrm{Ext}_{\overline{E}_\star(0)}(\overline{H}_\star, \overline{H}_\star)$, and from the Künneth theorem (e.g., [Wei94, Theorem 3.6.3]) we have $H(P_\bullet \otimes Q_\bullet) \cong H(P_\bullet) \otimes H(Q_\bullet)$, since homology of each of the complexes are free over $\overline{H}_\star$.

Another option for computing Ext over $\overline{E}_\star(0)$ and $\overline{E}_\star(1)$ is to use the topological realization map (Remark 1.7.14) and reduce to the computation in topology ([Rav86, Chapter 3]). This is done in [Orm11].

The Hopf algebroid $\overline{A}_\star(1)$ is not simple enough for direct computation with the cobar complex. The topological technique of computation is to use the Cartan-Eilenberg spectral sequence. However, this seems to break down when we have $\mathbb{Z}/2[\tau]$ in place of $\mathbb{Z}/2$. Instead there have been constructed explicit resolutions for $\overline{H}_\star$ over $\overline{E}_\star(0)$. Shkembi has done so in her thesis [Shk09]. We refer to [Hil11, Proposition 2.6] for the description of $\mathrm{Ext}_{\overline{A}_\star(1)}(\overline{H}_\star, \overline{H}_\star)$ given above. □

As a corollary of the above theorem and Equation (4.1) we have obtained the E_2-page for $q \equiv 1\ (4)$.

4.1.2 $q \equiv 3\ (4)$

In this case the class of ρ is nontrivial and $A_\star$ is not free over $\overline{A}_\star$ (or $\overline{H}_\star$) anymore. To pass this obstacle we use the ρ-Bockstein spectral sequence of Section 3.3 for the cobar complex. Denote this filtration by $F^{s+1} \subset F^s$. Since $\rho^2 = 0$, the filtration on the cobar complex is only nozero for $s = 0$ and $s = 1$. The associated graded algebra of this filtration is only nonzero in two grades, 0 and 1 (forgetting the other gradings for a moment). Since $\rho B \cong \overline{B} \cong B/\rho B$ and likewise for $H_\star$. We get for the associated graded algebra,

$$H(F^0/F^1) = \mathrm{Ext}_{\overline{B}}(\overline{H}_\star, \overline{H}_\star), \qquad H(F^1/F^2) = \mathrm{Ext}_{\overline{B}}(\overline{H}_\star, \overline{H}_\star)\{\rho\}.$$

That is, the E_1-page of the ρ-Bockstein spectral sequence is Theorem 3.3.1.

$$E_1 = \mathrm{Ext}_{\overline{B}}(\overline{H}_\star, \overline{H}_\star)[\rho]/\rho^2 \implies \mathrm{Ext}_B(H_\star, H_\star).$$

Here ρ has quadruple degree $(1, -1, 0-\alpha)$. From the description of the dual Steenrod algebra Proposition 1.7.3 and Equation (A.1), we observe that all of the differentials on generators in the cobar complex which do not involve τ, does not involve ρ. Hence if their differentials are zero modulo ρ, they are in fact zero. From this, and from degree reasons, we see that the only possible nonzero differential on a generator is on τ, which by Theorem 3.3.1 is $d_1(\tau) = \eta_L(\tau) - \eta_R(\tau) = \rho\tau_0$. Since the filtration ends in filtration degree 2, all higher differentials are zero, and the ρ-Bockstein spectral sequence collapse here.

Recall the formulas given for $\mathrm{Ext}_{\overline{B}}(\overline{H}_\star, \overline{H}_\star)$ above in Theorem 4.1.2. This gives us the E_1-page of the ρ-Bockstein spectral sequence. To calculate the E_2-page it is useful to introduce the redundant generator $[\rho\tau]$ and the relation $[\rho\tau] = \rho\tau$ to help with the bookkeeping. As the notation indicates, $[\rho\tau]$ will become indecomposable on the E_2-page. The computation is summarized in Theorem 4.1.3, Theorem 4.1.4 and Theorem 4.1.5 below. All of the generators have the grading defined in Theorem 4.1.2 above. Generators of the form $[\rho\tau]$ have degree equal to the sum of the degrees of their multiples.

Theorem 4.1.3. *For $H\mathbb{Z}_{(2)}$ we have:*

$$E_1 = \mathbb{Z}/2[\rho, \tau, h_0, [\rho\tau]]/(\rho^2, [\rho\tau] - \rho\tau),$$
$$E_2 = \mathbb{Z}/2[\rho, \tau^2, h_0, [\rho\tau]]/(\rho^2, \rho h_0, \rho[\rho\tau], [\rho\tau]^2).$$

The E_2-page is the E_∞-page, and the abutment is:

$$\mathrm{Ext}_{E_\star(0)}(H_\star, H_\star) = \mathbb{Z}/2[\rho, \tau^2, \tau_0, [\rho\tau]]/(\rho^2, \rho h_0, \rho[\rho\tau], [\rho\tau]^2).$$

Proof. The description of the E_2-page follows from the description of the differential. Since all higher differentials are zero, the E_2-page is the E_∞-page. It remains to show that there are no hidden multiplicative extensions, i.e., the multiplicative structure is the same as on the E_2-page. This follows from degree reasons. There is never an element at $(0, s, m+n\alpha)$ and $(1, s-1, m+n\alpha)$. All the elements in the E_∞-page are of the form:

generator	grading
$(\tau^2)^i h_0^j$	$(0, j, j+2i-2i\alpha)$
$\rho(\tau^2)^i h_0^j$	$(1, j-1, j+2i-(2i+1)\alpha)$
$[\rho\tau](\tau^2)^i h_0^j$	$(1, j-1, j+1+2i-(2i+2)\alpha)$

A multiplicative extension must lie below a multiple of ρ, i.e., $xy = 0\ (\rho)$. Since there are no elements of the form $(1, j-1, j+2i-2i\alpha)$, there are no multiplicative extensions. ☐

Theorem 4.1.4. *For kgl we have:*

$$E_1 = \mathbb{Z}/2[\rho, \tau, \tau_0, \tau_1, [\rho\tau]]/(\rho^2, [\rho\tau] - \rho\tau),$$
$$E_2 = \mathbb{Z}/2[\rho, \tau^2, \tau_0, \tau_1, [\rho\tau]]/(\rho^2, \rho\tau_0, \rho[\rho\tau], [\rho\tau]^2).$$

The E_2-page is the E_∞-page, and the abutment is:

$$\mathrm{Ext}_{E_\star(1)}(H_\star, H_\star) = \mathbb{Z}/2[\rho, \tau^2, h_0, \tau_1, [\rho\tau]]/(\rho^2, \rho h_0, \rho[\rho\tau], [\rho\tau]^2).$$

Proof. The description of the E_∞-page of kgl follows from the same argument as for $H\mathbb{Z}_{(2)}$. It remains to eliminate the possibility of hidden multiplicative extensions. Yet again this is impossible for degree reasons. All the elements in the E_∞-page are of the form:

generator	grading
$(\tau^2)^i h_0^j h_1^k$	$(0, j+k, j++2k+2i+(k-2i)\alpha)$
$\rho(\tau^2)^i h_0^j h_1^k$	$(1, j+k-1, j+2k+2i+(k-2i-1)\alpha)$
$[\rho\tau](\tau^2)^i h_0^j h_1^k$	$(1, j+k-1, j+2k+1+2i+(k-2i-2)\alpha)$

Assume that there is an element in the filtration degree above $(\tau^2)^i h_0^j h_1^k$. This element would have degree

$$(1, b+c-1, b+2c+2a+(c-2a-1)\alpha) \quad \text{or} \quad (1, b+c-1, b+2c+1+2a+(c-2a-2)\alpha),$$

for some nonnegative integers a, b and c. If we compare the degrees, these equations imply separately

$$c-k \equiv 1\ (2) \quad \text{and} \quad c-k \equiv 0\ (2).$$

This is impossible, hence, there are no multiplicative extensions. □

For ko there are considerably more generators and relations, and we introduce one additional redundant relation on the E_1-page, $[h_1\tau] = h_1\tau$. In addition, there are hidden multiplicative extensions when passing from the E_∞-page to the abutment.

Theorem 4.1.5. *For ko we have:*

$$E_1 = \frac{\mathbb{Z}/2[\tau, h_0, h_1, v, w, [h_1\tau], [\rho\tau]]}{h_0h_1, \tau h_1^3, h_1v, v^2 - h_0^2w, \rho^2, [h_1\tau] - h_1\tau, [\rho\tau] - \rho\tau},$$

$$E_2 = \frac{\mathbb{Z}/2[\tau^2, h_0, h_1, v, w, [h_1\tau], [\rho\tau]]}{\begin{array}{c} h_0h_1, \tau^2h_1^3, h_1v, v^2 - h_0^2w, \rho h_0, \rho^2, \\ h_0[h_1\tau], h_1^2[h_1\tau], v[h_1\tau], [h_1\tau]^2 - \tau^2h_0^2, \\ [\rho\tau]h_1^3, \rho[h_1\tau] = h_1[\rho\tau], \rho[\rho\tau], [\rho\tau]^2, [h_1\tau][\rho\tau] = \rho\tau^2h_1 \end{array}}.$$

The E_2-page is the E_∞-page, and the abutment is:

$$\mathrm{Ext}_{A_\star(1)}(H_\star, H_\star) = \frac{\mathbb{Z}/2[\tau^2, h_0, h_1, v, w, [h_1\tau], [\rho\tau]]}{\begin{array}{c} h_0h_1, \tau^2h_1^3 = v\rho, h_1v, v^2 - h_0^2w, \rho h_0, \rho^2, \\ \rho h_1[h_1\tau] = h_0[h_1\tau], h_1^2[h_1\tau], v[h_1\tau], [h_1\tau]^2 - \tau^2h_0^2, \\ [\rho\tau]h_1^3, \rho[h_1\tau] = h_1[\rho\tau], \rho[\rho\tau], [\rho\tau]^2, [h_1\tau][\rho\tau] = \rho\tau^2h_1 \end{array}}.$$

Proof. The description of the E_2-page follows from the same argument as for $H\mathbb{Z}_{(2)}$. The multitude of new relations arise from the relations on the E_1-page in which h_1, ρ and τ take part. With this in mind, the algebra is not really as complicated as it appears. Now we have to go hunting for the hidden multiplicative extensions. To make it easier to follow the argument, the reader might want to look ahead at Figure 4.4 on page 60 which displays the final answer. This figure is simultaneously a picture of the E_∞-page and $\mathrm{Ext}_{A_\star(1)}(H_\star, H_\star)$. The conventions used when drawing the figures are explained in Section 4.5.

To determine the possible hidden multiplicative extensions we consider Massey products. The following computation is similar to the computation in [Hil11, Proposition 4.3]. By considering the elements in the various degrees, we obtain the following table for elements on the E_2-page, in Ext, and their representatives in the cobar complex:

E_2 and Ext	Cobar
h_0	τ_0
h_1	ξ_1
$[h_1\tau]$	τ_0^2

It is easy to check in the cobar complex, that we have the differentials

$$
\begin{aligned}
d([\tau]) &= [\rho\tau_0], \\
d([\tau_1]) &= [\xi_1|\tau_0], \\
d([\tau_1 + \xi_1\tau_0]) &= [\tau_0|\xi_1], \\
d([\tau_0\tau_1]) &= [\tau_1 + \xi_1\tau_0|\tau_0] + [\tau_0|\tau_1] + [\xi_1|\tau_0^2].
\end{aligned}
$$

One easy consquence is that $h_0h_1 = 0$ in Ext, since $[\xi_1|\tau_0] = d([\tau_1])$. We can also compute that the Massey product $\langle \rho, h_0, h_1 \rangle$ is

$$\langle \rho, h_0, h_1 \rangle = \rho[\tau_1 + \xi_1\tau_0] + \tau[\xi_1] = [\rho\tau_1 + \rho\xi_1\tau_0 + \tau\xi_1] = [\tau_0^2].$$

Using the juggling result, Lemma A.5.2, we get

$$[h_1\tau]h_0 = \langle \rho, h_0, h_1 \rangle h_0 = \rho\langle h_0, h_1, h_0 \rangle.$$

The last Massey product is

$$\langle h_0, h_1, h_0 \rangle = [\tau_1 + \xi_1\tau_0|\tau_0] + [\tau_0|\tau_1] = [\xi_1|\tau_0^2] + d([\tau_0\tau_1]).$$

The indeterminacy is zero for degree reasons. Hence we have found a hidden multiplicative extension in Ext. We have $[h_1\tau]h_0 = \rho h_1[h_1\tau]$.

By a similar computation performed in [Hil11, Proposition 4.3] there is a relation $\tau^2h_1^3 = v\rho$. By inspection of Figure 4.4 these generate all the possible multiplicative extensions. This follows from the commutativity and associativity of the algebra. □

We have now determined the E_2-page for the Adams spectral sequence when $q \equiv 1$ (4) and $q \equiv 3$ (4) for $H\mathbb{Z}_{(2)}$, kgl and ko. For $H\mathbb{Z}_{(2)}$ and ko they are drawn in Figure 4.1 on page 58, Figure 4.2 on page 58, Figure 4.3 on page 59 and Figure 4.4 on page 60.

Remark 4.1.6. Because of our choice of grading in Section 4.5, the d_r-differential goes one square to the left and r steps out of the page. Hence, from the pictures we conclude that most differentials are nonzero (remember, this is a spectral sequence of algebras, we only have to consider differentials on generators and use the Leibniz rule). From this we conclude that only powers of τ support differentials. We now proceed to determine these.

4.2 Differentials

The abutment of the Adams spectral sequence for $H\mathbb{Z}_{(2)}$ is the 2-completed homotopy groups $\pi_\star(H\mathbb{Z}_{(2)})_{\hat{2}} = \pi_\star H\mathbb{Z}_2 = H_\star(\operatorname{Spec}\mathbb{F}_q; \mathbb{Z}_2)$. These groups are known from Example 1.2.7,

$$H_{m+n\alpha}(\operatorname{Spec}\mathbb{F}_q; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & m = n = 0, \\ \mathbb{Z}/2^{\nu_2(q^i-1)} & m = i-1,\ n = -i,\ i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

This information determines the differentials in the Adams spectral sequence for $H\mathbb{Z}_{(2)}$. The maps $ko \to kgl \to H\mathbb{Z}_{(2)}$ then determine the differentials in the Adams spectral sequence for kgl and ko.

Lemma 4.2.1. *When $q \equiv 1$ (4) the following is true for the Adams spectral sequence of $H\mathbb{Z}_{(2)}$.*

- *τ^{2^s} survives to the $E_{\nu_2(q-1)+s}$-page and is killed off by a $d_{\nu_2(q-1)+s}$-differential.*
- *Terms of the form $\rho h_0^i\tau^{k-1}$ are zero for $i > \nu_2(k)$ on the E_∞-page.*

- *The only supported differentials are*

$$d_{\nu_2(q-1)+\nu_2(k)}(h_0^i\tau^k) = \rho h_0^{i+\nu_2(q-1)+\nu_2(k)}\tau^{k-1},\ k \geq 1,$$

or more briefly on algebra generators on the $E_{\nu_2(q-1)+s}$-page

$$d_{\nu_2(q-1)+s}(\tau^{2^s}) = \rho\tau^{2^s-1}h_0^{\nu_2(q-1)+s},\ s \geq 0.$$

Proof. For degree reasons neither h_0 nor ρ can support any differentials or be the targets of any differentials. Hence, they are infinite cycles, and a differential on an element simplifies:

$$d_r(\rho^\epsilon h_0^i\tau^k) = \rho^\epsilon h_0^i d_r(\tau^k).$$

Since the only possible target for $d_r(\tau^k)$ is $\rho h_0^r\tau^{k-1}$, there can be no differential on terms with $\epsilon = 1$. Observe that if τ^k supports a d_r-differential, then it will leave a h_0-tower of height r to the left. With this observation and the knowledge of the abutment we can read off the differentials directly. The first supported d_r-differential on τ must be for $r = \nu_2(q-1)$. Any earlier differential would leave the wrong groups in the abutment. From the Lebiniz rule all powers of τ^s with $\nu_2(s) = 0$ are killed off by this differential, leaving powers of τ with $\nu_2(s) = 1$. Continuing like this we get by induction on the $E_{\nu_2(q-1)+s}$-page:

- All terms of the form $h_0^i\tau^k$ for $\nu_2(k) < s$ are zero.
- All terms of the form $\rho h_0^i\tau^{k-1}$, with $\nu_2(k) < s$ and $i > \nu_2(k)$ are zero.
- There is a differential on τ^{2^s} as prescribed.

□

Lemma 4.2.2. *When $q \equiv 3\ (4)$ the following is true for the E_2-page of the Adams spectral sequence of $H\mathbb{Z}_{(2)}$.*

- *τ^{2^s} survives to the $E_{\nu_2(q^2-1)+s-1}$-page and is killed off by a $d_{\nu_2(q^2-1)+s-1}$-differential.*
- *Terms of the form $\rho h_0^i\tau^{k-1}$ are zero for $i > \nu_2(k)$ on the E_∞-page.*
- *The only supported differentials are*

$$d_{\nu_2(q^2-1)+\nu_2(k)}(h_0^i\tau^{2k}) = \rho h_0^{i+\nu_2(q^2-1)+\nu_2(2k)}\tau^{2k-1},\ k \geq 1$$

or more briefly on algebra generators on the $E_{\nu_2(q^2-1)+s}$-page

$$d_{\nu_2(q^2-1)+s-1}(\tau^{2^s}) = \rho\tau^{2^s-1}h_0^{\nu_2(q^2-1)+s},\ s \geq 1.$$

In this listing it is implicit that $[\rho\tau] = \rho\tau$.

Proof. The proof is essentially the same as the proof of Lemma 4.2.1. For degree reasons h_0, ρ and $[\rho\tau]$ are infinite cycles. A differential on an element simplifies:

$$d_r(\rho^{\epsilon_1}[\rho\tau]^{\epsilon_2}h_0^i\tau^k) = \rho^{\epsilon_1}[\rho\tau]^{\epsilon_2}h_0^i d_r(\tau^k).$$

Since the only possible target for $d_r(\tau^k)$ is $[\rho\tau]h_0^r\tau^{k-2}$, there can be no differential on terms with $\epsilon_1 = 1$ or $\epsilon_2 = 1$. Observe that if τ^k supports a d_r-differential, then it will leave a h_0-tower of height r to the left. With this observation and the knowledge of the abutment we can read off the differentials directly. The first supported d_r-differential on τ^2 must be for $r = \nu_2(q^2-1)$. Any earlier differential would leave the wrong groups in the abutment. From the Lebiniz rule all powers of τ^s with $\nu_2(s) = 1$ are killed off by this differential, leaving powers of τ with $\nu_2(s) = 2$. Continuing like this we get on the $E_{\nu_2(q^2-1)+s-1}$-page:

- All terms of the form $h_0^i\tau^{2k}$ for $\nu_2(2k) < s$ are zero.

- All terms of the form $[\rho\tau]h_0^i\tau^{2k-2}$, with $\nu_2(2k) < s$ and $i > \nu_2(2k)$ are zero.

- There is a differential on τ^{2^s} as prescribed.

$\square$

Having determined the differentials in the Adams spectral sequence for $H\mathbb{Z}_2$ we proceed to determine the differentials in the Adams spectral sequences for $H\mathbb{Z}_{(2)}$ and ko. From Equation (1.9) and Equation (1.6) we are provided with maps

$$ko \to kgl \to H\mathbb{Z}_{(2)}.$$

We consider the map of spectral sequences induced by $ko \to H\mathbb{Z}_{(2)}$. On the E_2-page, this is the map induced by $H_\star ko \to H_\star kgl \to H_\star H\mathbb{Z}_{(2)}$ on $\mathrm{Ext}_{A_\star}(H_\star, H_\star ko) \to \mathrm{Ext}_{A_\star}(H_\star, H_\star H\mathbb{Z}_{(2)})$. Recall the identification of the E_2-pages in Theorem 4.1.2 and Theorem 4.1.5. Through this map, the elements corresponding to τ, τ^2, h_0 and $[\rho\tau]$ map to the corresponding elements in $E_2(H\mathbb{Z}_{(2)})$. This is because they are the only elements in their respective degrees, and in the cobar complex they are represented by ρ, τ and τ_0, which map isomorphically between the cobar complexes. From Remark 4.1.6 powers of τ determine the differentials. Let $f_r : E_r \to E_r$ denote the map on the E_r-pages. Since $d_r f_r = f_r d_r$, we get the same differentials on the power of τ in $E_r(ko)$ as in $E_r(H\mathbb{Z}_{(2)})$. That is, they have the form described in Lemma 4.2.1 and Lemma 4.2.2. It is now straightforward to compute the E_∞-page of both kgl and ko.

4.3 The E_∞-page

The E_∞-page for ko is displayed in Figure 4.5 on page 61 and Figure 4.6 on page 62 when $q \equiv 1$ (4) and $q \equiv 3$ (4). We give explicit descriptions of the E_∞-pages in Theorem 4.3.1 and Theorem 4.3.2 below, however, the figures contain the best description.

Theorem 4.3.1. *If $q \equiv 1$ (4) then the E_∞-page for the Adams spectral sequence for ko is $D[w, u]$, where D is additively*

$$D := \begin{cases} \mathbb{Z}/2[h_0]\{1\} & |1| = (0, 0 + 0\alpha), \\ \mathbb{Z}/2[h_0]\{v\} & |v| = (3, 5 + 2\alpha), \\ \mathbb{Z}/2\{h_1^i\} & |h_1^i| = (i, i + i\alpha),\ i \geq 1, \\ \mathbb{Z}/2[h_0]/(h_0^{\nu_2(q-1)+\nu_2(i+1)})\{u\tau^i\} & |u\tau^i| = (0, i - (1+i)\alpha),\ i \geq 0, \\ \mathbb{Z}/2[h_0]/(h_0^{\nu_2(q-1)+\nu_2(i+1)})\{uv\tau^i\} & |uv\tau^i| = (3, 5 + i + (1-i)\alpha),\ i \geq 0, \\ \mathbb{Z}/2\{\tau^i h_1^2\} & |\tau^i h_1^2| = (2, i + 2 + (2-i)\alpha),\ i \geq 1, \\ \mathbb{Z}/2\{\tau^i h_1\} & |\tau^i h_1| = (1, i + 1 + (1-i)\alpha),\ i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here the generators of $D[w, u]$ are subject to the multiplicative relations induced from the E_2-page. In particular any occurrence of u^2 is zero (e.g., $u\{u\tau^i\} = 0$).

Theorem 4.3.2. *If $q \equiv 3$ (4) then the E_∞-page for the Adams spectral sequence for ko is $D[w, \rho]$,* *where D is*

$$D := \begin{cases} \mathbb{Z}/2[h_0]\{1\} & |1| = (0, 0+0\alpha), \\ \mathbb{Z}/2[h_0]\{v\} & |v| = (3, 5+2\alpha), \\ \mathbb{Z}/2\{h_1^i\} & |h_1^i| = (i, i+i\alpha),\ i \geq 1, \\ \mathbb{Z}/2\{\rho\tau^i\} & |\rho\tau^i| = (0, i-(1+i)\alpha),\ i \geq 0,\ i \text{ even}, \\ \mathbb{Z}/2[h_0]/(h_0^{\nu_2(q^2-1)-1+\nu_2(i+1)})\{\rho\tau^i\} & |\rho\tau^i| = (0, i-(1+i)\alpha),\ i \geq 0,\ i \text{ odd}, \\ \mathbb{Z}/2\{\rho v\tau^i\} & |\rho v\tau^i| = (3, i+5+(1-i)\alpha),\ i \geq 0,\ i \text{ even}, \\ \mathbb{Z}/2[h_0]/(h_0^{\nu_2(q^2-1)-1+\nu_2(i+1)})\{\rho v\tau^i\} & |\rho v\tau^i| = (3, i+5+(1-i)\alpha),\ i \geq 0,\ i \text{ odd}, \\ \mathbb{Z}/2\{h_1^2\tau^i\} & |\tau^i h_1^2| = (2, i+2+(2-i)\alpha),\ i \geq 1, \\ \mathbb{Z}/2\{h_1\tau^i\} & |\tau^i h_1| = (1, i+1+(1-i)\alpha),\ i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here the generators of $D[w, \rho]$ are subject to the multiplicative relations induced from the E_2-page. In particular any occurrence of ρ^2 is zero.

4.4 The Abutment

We are now ready to compute the abutment. To do so we need to know what represents h_0 in $\pi_0 ko$. The description of $\pi_0 S$ by Morel provides us with this information.

Theorem 4.4.1 ([Mor04], [Hil11, Lemma 5.4])**.** *The class h_0 is represented by $2-\rho\eta$ in $\pi_{0+0\alpha}ko$. Here $\rho \in \pi_{-\alpha}ko$ and η is the Hopf map $\eta \in \pi_\alpha ko$. The class h_1 is represented by $\eta \in \pi_\alpha ko$.*

Recall that each filtration is contained in a single box in the figures. Hence, we essentially only have to worry about the h_0-towers. To save some space we follow [RW00, Proposition 1.9] and define w_i to be

$$w_i = \begin{cases} 2^{\nu_2(q-1)+\nu_2(i)} & q \equiv 1\ (4), \\ 2^{\nu_2(q^2-1)-1+\nu_2(i)} & q \equiv 3\ (4),\ i \text{ even}, \\ 2 & q \equiv 3\ (4),\ i \text{ odd}. \end{cases}$$

Theorem 4.4.2. *Let $z = 4(1+\alpha)$. When $q \equiv 1\ (4)$ the abutment is, for $k \geq 0$,*

$$\pi_\star ko = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}/2 & kz, \\ \mathbb{Z}_2 & 2+2\alpha+kz, \\ (\mathbb{Z}/2)^2 & i\alpha + kz,\ i \geq 0, \\ \mathbb{Z}/w_{i+1} & i-(1+i)\alpha+kz,\ i \geq 1, \\ \mathbb{Z}/w_{i+1} & i+2+(1-i)\alpha+kz,\ i \geq 1, \\ \mathbb{Z}/2 & i+(2-i)\alpha+kz,\ i \geq 1, \\ (\mathbb{Z}/2)^2 & i+(1-i)\alpha+kz,\ i \geq 1, \\ \mathbb{Z}/2 & i-i\alpha+kz,\ i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. In this case $\rho = 0$, so h_0 is represented by 2, and the arguments for reading off the convergence are relatively standard when using Equation (3.1). The abutment can be read directly off from the E_∞-page.

Let A denote the abutment in a specific bidegree, and let the filtration be $A = F^0 \supset F^1 \supset \cdots$ (this does not necessarily correspond to the filtration grading in the spectral sequence, a more precise argument would follow by a straightforward change of indices). We have to prove that

1. The groups $\mathbb{Z}/2[h_0]/h_0^n$ each become copies of $\mathbb{Z}/2^n$.

2. $\mathbb{Z}/2[h_0]$ become a $\mathbb{Z}_2$,

3. The groups in dimensions $i + (1-i)\alpha$ each become a $(\mathbb{Z}/2)^2$.

We start by proving case 1 and 2. Pick an element $a \in A$ such that the class $[a]$ in $\mathbb{Z}/2 = F^0/F^1$ is a generator. Since $h_0[a] \neq 0 \in \mathbb{Z}/2 = F^1/F^2$ we have $2a \neq 0$. Continuing like this, we get inductively that $2^i x \neq 0$, and $A/F^i = \mathbb{Z}/2^i$. In case 1, this ends after n steps, since $F^n = 0$. For case 2, consider the short exact sequence of inverse systems,

$$0 \to F^i \to A \to A/F^i \to 0.$$

Since the filtration is Hausdorff and complete, the lim-lim^1-sequence (e.g., [Wei94, Corollary 3.5.4]) implies $A = \lim_i A/F^i = \mathbb{Z}_2$.

The proof of case 3 is similar. We have to decide between $(\mathbb{Z}/2)^2$ and $\mathbb{Z}/4$. Consider an element $a \in A$ which is nonzero in F^0/F^1. Since $h_0[a] = 0$, we get $2a = 0$ modulo F^2. Hence, we must have the group $(\mathbb{Z}/2)^2$. This also explains why we get $\mathbb{Z}_2 \oplus \mathbb{Z}/2$ in degree 0. $\square$

Theorem 4.4.3. *Let $z = 4(1+\alpha)$. When $q \equiv 3$ (4) the abutment is, for $k \geq 0$,*

$$\pi_\star ko = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}/2 & kz, \\ \mathbb{Z}_2 & 2 + 2\alpha + kz, \\ \mathbb{Z}/4 & i\alpha + kz, i \geq 0, \\ \mathbb{Z}/w_{i+1} & i - (1+i)\alpha + kz,\ i \geq 0, \\ \mathbb{Z}/w_{i+1} & i + 2 + (1-i)\alpha + kz,\ i \geq 0,, \\ \mathbb{Z}/2 & i + (2-i)\alpha + kz,\ i \geq 1, \\ (\mathbb{Z}/2)^2 & i + (1-i)\alpha + kz,\ i \geq 1,\ i \text{ odd}, \\ \mathbb{Z}/4 & i + (1-i)\alpha + kz,\ i \geq 1,\ i \text{ even}, \\ \mathbb{Z}/2 & i - i\alpha + kz,\ i \geq 1, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. In this case h_0 is represented by $2 - \rho\eta$, and we have to be a bit more careful when determining the extensions. On the E_∞-page η represents h_1. Let A denote the abutment in a specific bidegree, and let the filtration be $A = F^0 \supset F^1 \supset \cdots$. We have to prove that

1. The groups $\mathbb{Z}[h_0]/h_0^n$ become copies of $\mathbb{Z}/2^n$.

2. The groups $\mathbb{Z}[h_0]$ become copies of $\mathbb{Z}_2$.

3. The groups in degree $i\alpha$ are $\mathbb{Z}/4$.

4. The groups in degree $i + (1-i)\alpha$ are $(\mathbb{Z}/2)^2$ for i odd and $\mathbb{Z}/4$ for i odd.

We begin by proving case 1 and 2. Pick an element $a \in A$ such that $[a]$ is a generator of $\mathbb{Z}/2 = F^0/F^1$. Then $h_0[a]$ is represented by $2a + \rho\eta a$. In all the degrees in which we have a group $\mathbb{Z}[h_0]/h_0^n\{[a]\}$, $\rho h_1[a]$ is zero. Hence, the calculation reduces to the one in Theorem 4.4.2. For case 2 this is not necessarily so, since in degree 0 we have a ρh_1. Instead we get that $(2 - \rho\eta)a = g_1$ modulo F^2, where g_1 is some class representing $h_0[a]$. Hence $2a = g + \rho\eta[a]$. Hence we have $A/F^2 = \mathbb{Z}/4 \oplus \mathbb{Z}/2\{\rho h_1[a]\}$. Multiplying with h_0 again we obtain $4a = g_2$ modulo F^3, where g_2 represents h_0^2, since $\rho\eta(2 + \rho\eta) = 0$. Hence we get inductively $A/F^n = \mathbb{Z}/2^n \oplus \mathbb{Z}/2$ for $n \geq 2$. From a lim-lim^1 argument as in Theorem 4.4.2 we obtain the required convergence.

The proof of case 3 and 4 are variants of this argument. Pick an element $a \in A$ such that it is a generator of $\mathbb{Z}/2 = F^0/F^1$. For the groups in degree $i + (1-i)\alpha$ when i is odd, there is a h_0 multiplication, giving $h_0[a] = \rho h_1[a]$, which implies $2a - \rho\eta a = \rho\eta a$, hence $2a = 0$, so we must have $(\mathbb{Z}/2)^2$ (the sign of $\rho\eta a$ does not matter since it is an element of order 2). When i is even or we are in case 3, we get a multiplication $h_0 a = 2a - \rho\eta a = 0$. Hence $2a = \rho\eta a \neq 0$, and we obtain $\mathbb{Z}/4$. $\square$

4.4.1 Comparison with Friedlander

From Equation (1.3) we have that $\pi_m KO = KO_m\mathbb{F}_q$ and $\pi_{m+2\alpha}KO = KSp_{m-2}\mathbb{F}_q$. These groups can be read off from Theorem 4.4.2 and Theorem 4.4.3. For $m \geq 0$ we get the table

m modulo 8	0	1	2	3	4	5	6	7
$\widetilde{KO}_m(\mathbb{F}_q)_{\hat{2}}$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$\mathbb{Z}/w_{(m+1)/2}$	0	0	0	$\mathbb{Z}/w_{(m+1)/2}$
$\widetilde{KSp}_m(\mathbb{F}_q)_{\hat{2}}$	0	0	0	$\mathbb{Z}/w_{(m+1)/2}$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$\mathbb{Z}/w_{(m+1)/2}$

Here $\widetilde{KO}_m$ and $\widetilde{KSp}_m$ are the reduced theories, i.e., there is a missing $\mathbb{Z}$ at $m = 0$. The table above is obtained by reading off the homotopy groups in weight 0. As an example, consider the contribution from $\mathbb{Z}/w_{i+1}$ in dimensions $i - (1+i)\alpha + 4k(1+\alpha)$. For this to be in weight zero we must have $4k - 1 = i$. Hence, $\pi_{8k-1}\mathbf{KO} = \mathbb{Z}/w_{i+1} = \mathbb{Z}/w_{8k/2}$, and we have obtained the last group in the top row with $m = 8k - 1$.

If we use Lemma B.1.3 we see that we get perfect agreement with [Fri76].

Proposition 4.4.4 ([Fri76, Theorem 1.7])**.** *Hermitian K-theory KO and Symplectic K-theory KSp of finite fields of odd characteristic are given in the following table.*

m modulo 8	$\widetilde{KO}_m(\mathbb{F}_q)$	$\widetilde{KSp}_m(\mathbb{F}_q)$
0	$\mathbb{Z}/2$	0
1	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	0
2	$\mathbb{Z}/2$	0
3	$\mathbb{Z}/q^{(m+1)/2} - 1$	$\mathbb{Z}/q^{(m+1)/2} - 1$
4	0	$\mathbb{Z}/2$
5	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
6	0	$\mathbb{Z}/2$
7	$\mathbb{Z}/q^{(m+1)/2} - 1$	$\mathbb{Z}/q^{(m+1)/2} - 1$

4.5 Images of the E_2- and E_∞-pages

An element in degree $(s, m + n\alpha)$ is drawn at $(m - s, n)$. For kgl the picture would look much the same as for $H\mathbb{Z}_{(2)}$, but with lines of τ_1 towers with slope 1 emanating from each generator in the uncluttered picture of $H\mathbb{Z}_{(2)}$. The d_r-differential goes one square to the left and r steps out of the page. The choice of grading is such that the filtration quotients of the abutment are contained in each box. Hence, the E_∞-page is almost a picture of the abutment.

We make the definition $x = [h_1\tau]$ and $y = [\rho\tau]$. The blue and teal dots are infinite towers on h_0. The red dots are a single copy of that generator. The purple dots represent a single generator, but support a h_0 multiplication. In the E_∞-pages the black dots represents groups where some generators have been killed off. The height of the tower is indicated next to the dot. We set $a = \nu_2(q^2 - 1) - 1$.

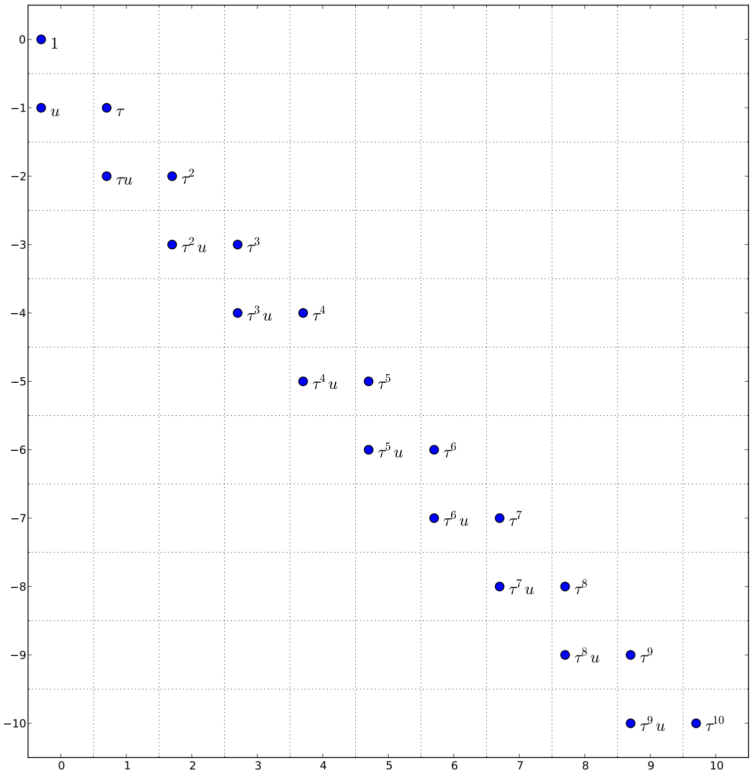


Figure 4.1: E_2-page for $q \equiv 1$ (4) for $H\mathbb{Z}_{(2)}$. The blue dots are infinite towers on h_0.

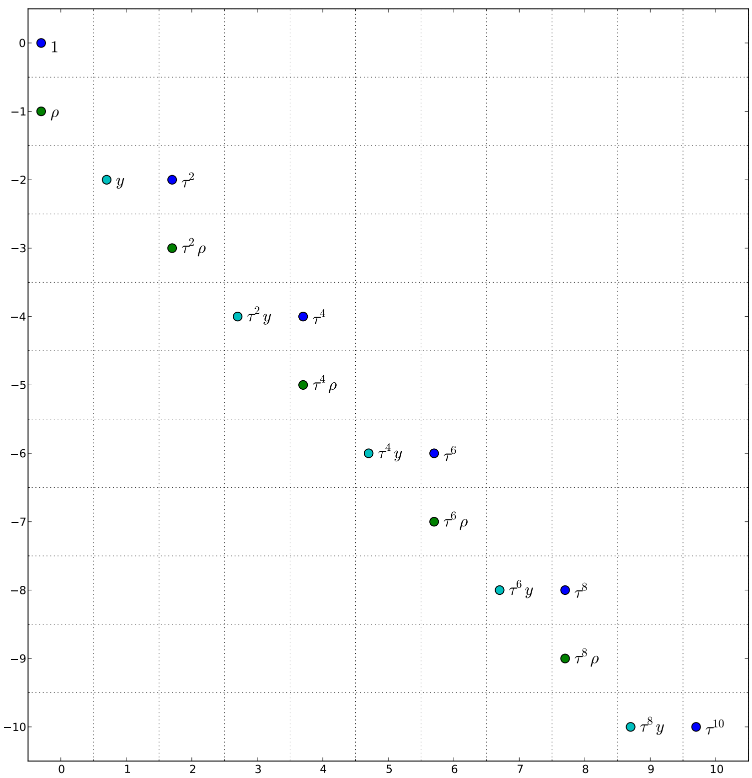


Figure 4.2: E_2-page for $q \equiv 3$ (4) for $H\mathbb{Z}_{(2)}$. Blue and teal dots are infinite towers on h_0. Green dots are a copy of $\mathbb{Z}/2$. Here $x = [h_1\tau]$ and $y = [\rho\tau]$.

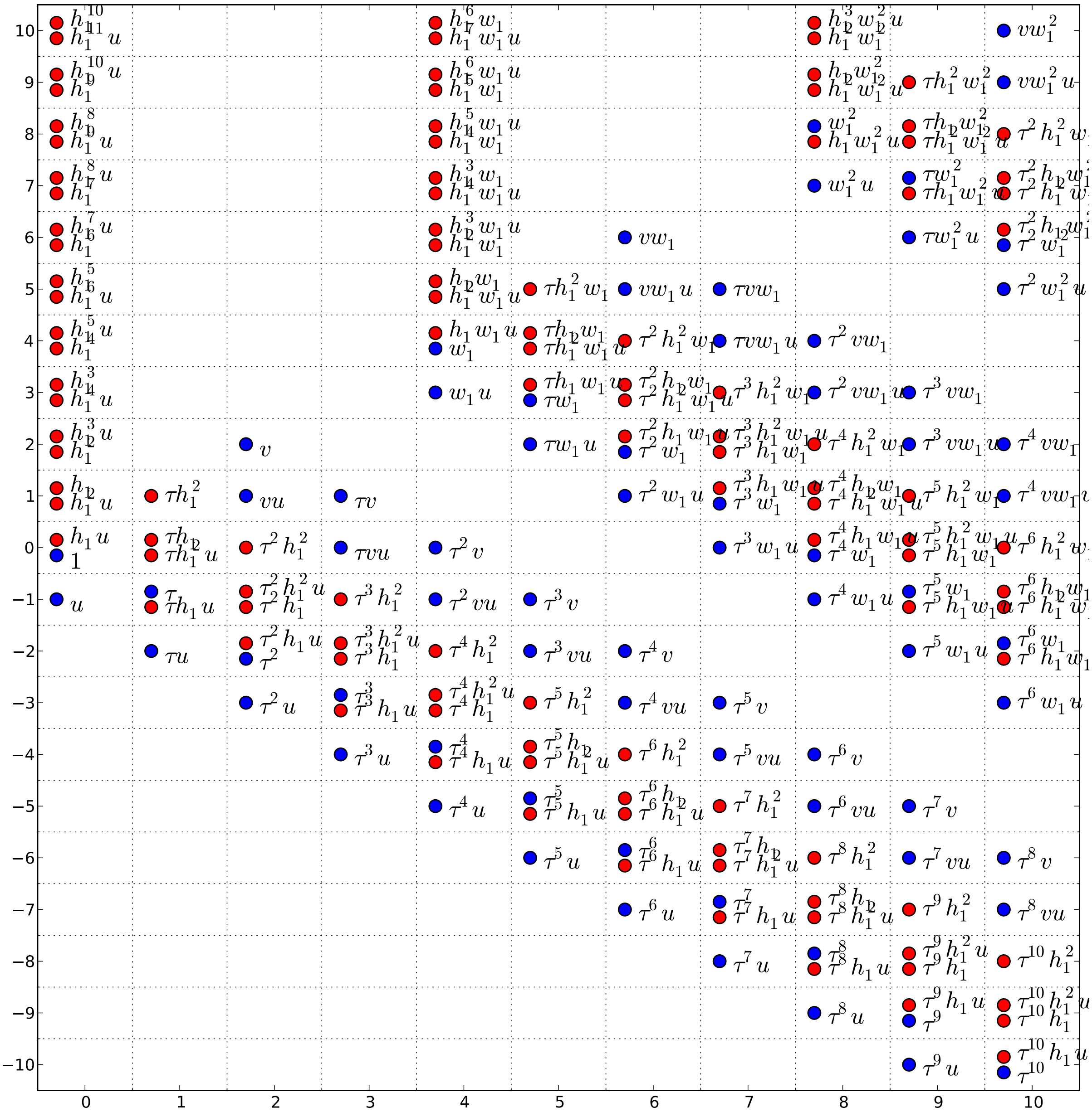


Figure 4.3: E_2-page for $q \equiv 1$ (4) for ko. Red dots are copies of $\mathbb{Z}/2$. Blue dots are infinite towers on h_0. In the E_∞-page all blue dots with a blue dot to the left disappear. Those remaining have a tower of $\mathbb{Z}/2$'s of height $\nu_2(q-1) + \nu_2(\text{power of } \tau \text{ to the right})$.

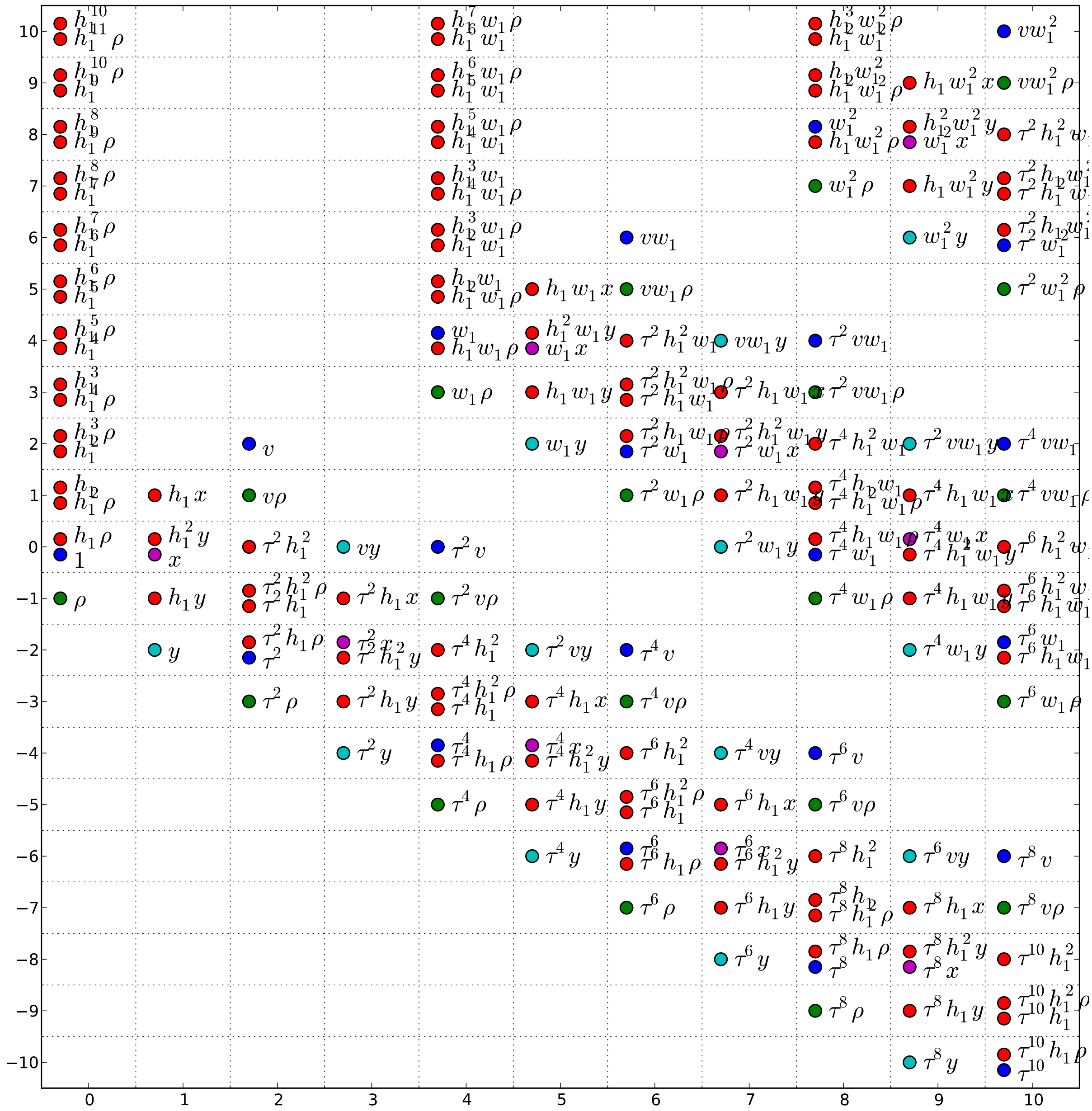


Figure 4.4: E_2-page for $q \equiv 3$ (4) for ko. Red, green and purple dots are copies of $\mathbb{Z}/2$. Blue and teal dots are infinite towers on h_0. Remember that $x = h_1\tau$ and $y = \rho\tau$. The purple dots support multiplication by h_0. In the E_∞-page almost all blue dots disappear, while the teal dots have a tower of $\mathbb{Z}/2$'s of height $\nu_2(q+1) + \nu_2$(power of τ to the right).

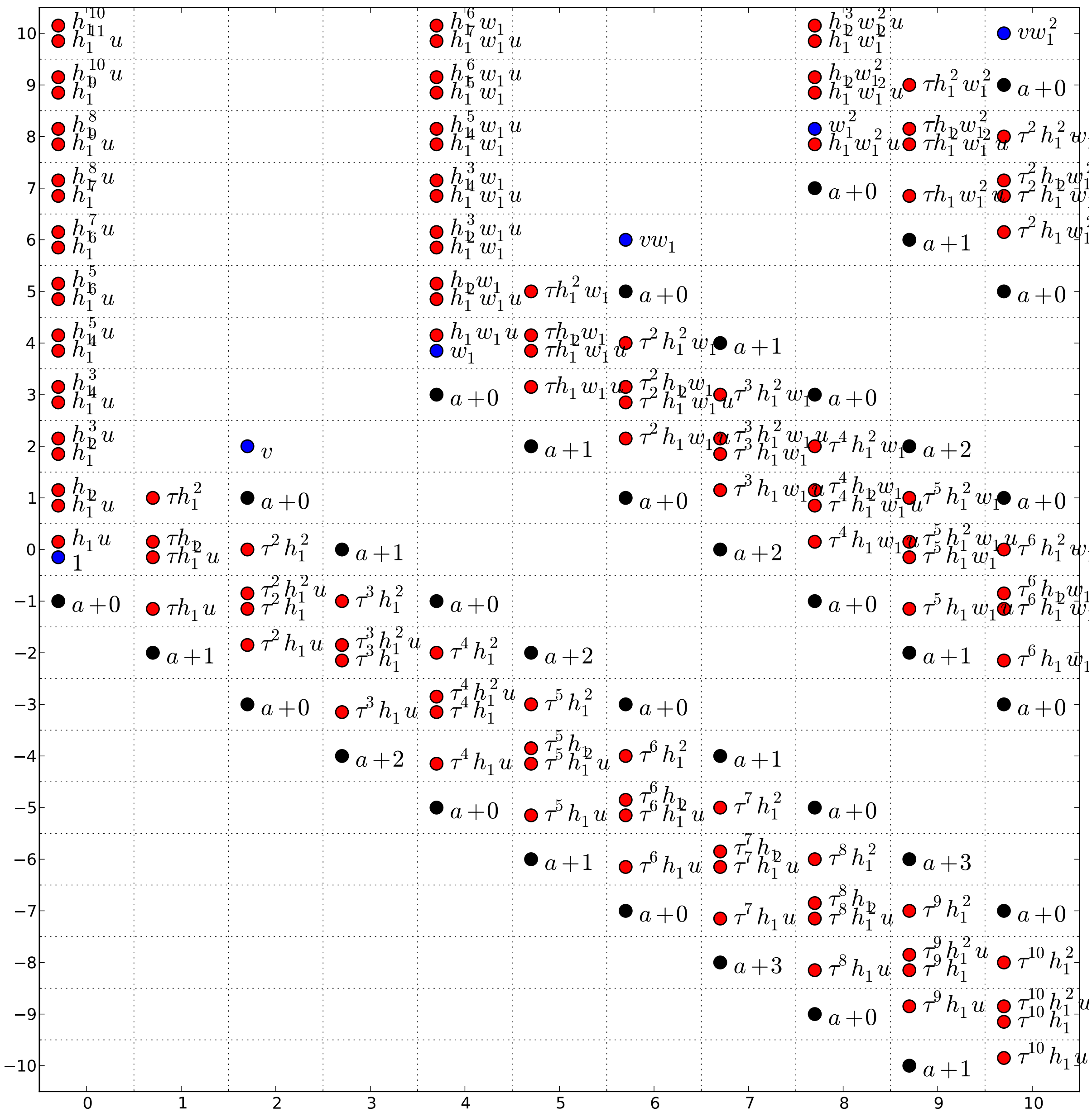


Figure 4.5: E_∞-page for $q \equiv 1$ (4) for ko. Red dots are copies of $\mathbb{Z}/2$. Blue dots are infinite towers on h_0. The black dots have a h_0-tower of height $a+i$, where $a = \nu_2(q-1)$.

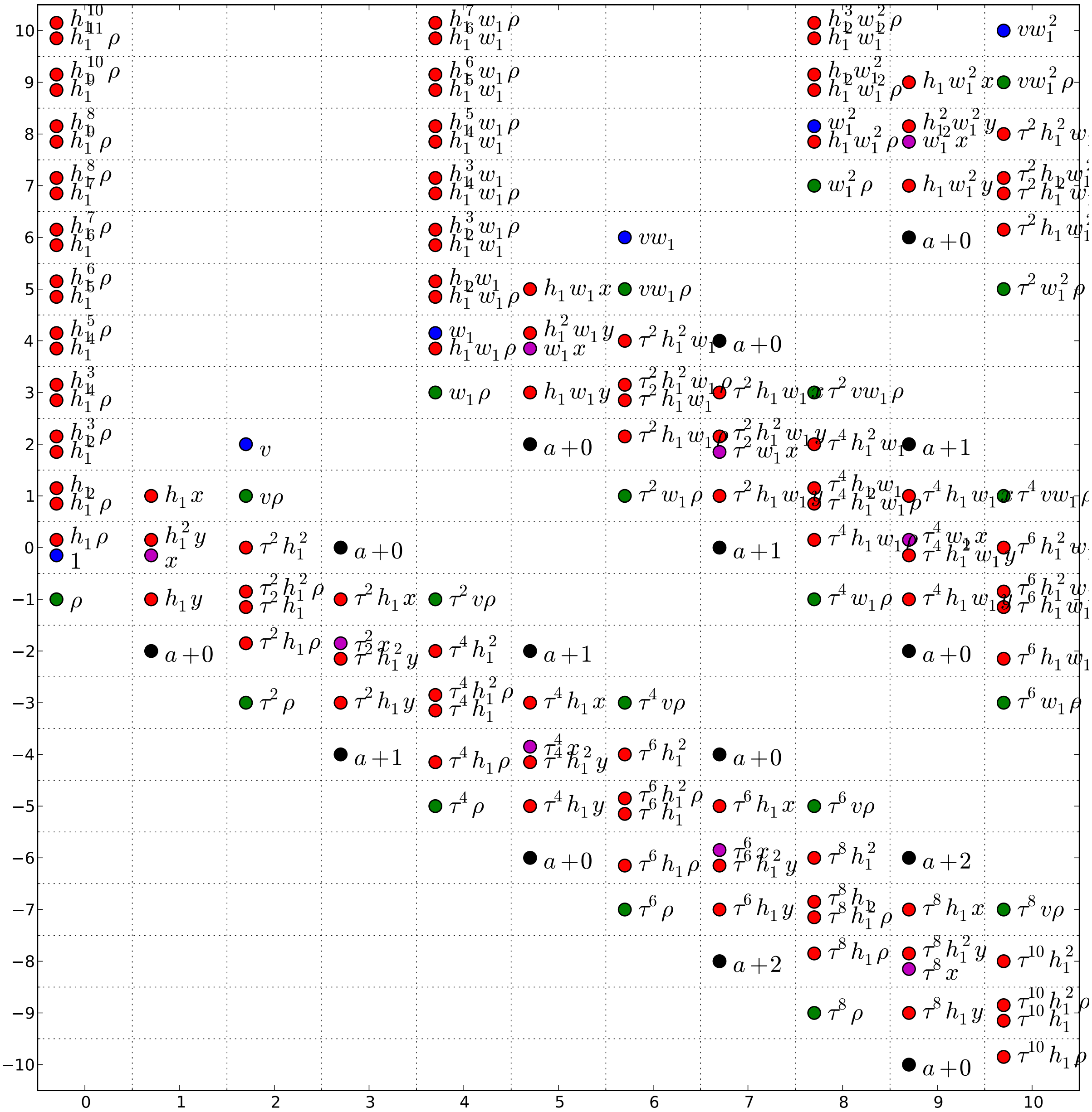


Figure 4.6: E_∞-page for $q \equiv 3$ (4) for ko. Red, green and purple dots are copies of $\mathbb{Z}/2$. Blue dots are infinite towers on h_0. The purple dots support multiplication by h_0. The black dots have a h_0-tower of height $a+i$, where $a = \nu_2(q^2-1)-1$.

A Hopf Algebroids

In this appendix we recall some of the properties of Hopf algebroids and comodules which are needed elsewhere in the text. The definitions are taken from [Rav86, Appendix A]. We will use some standard homological algebra, a good reference is [Wei94].

Before we start we would like to make some conventions. In this section K denotes a commutative ring with unit. In the rest of the thesis K is $\mathbb{Z}/2$, so signs are not particularly important. We make the convention that $\otimes = \otimes_K$ when tensoring objects. When we consider elements of tensor products, or tensor products of functions we leave the index out, since this can be determined from the domains. An object and its identity morphism are occasionally denoted by the same symbol.

The objects we work with are bigraded K-modules, but for most of the constructions one can simply forget about the grading.

A.1 Bigraded Modules, Hopf Algebroids and Comodules

We briefly recall some basic properties of the category of bigraded algebras.

- The category of bigraded K-modules ${}_K\mathrm{Mod}$ has as objects sets of K-modules $\{M_{n,m}\}_{n,m\in\mathbb{Z}}$. The morphisms $f : M \to N$ are sets of K-linear maps $f_{n,m} : M_{n,m} \to N_{n,m}, n, m \in \mathbb{Z}$.
- The tensor product of bigraded K-modules M, N is the bigraded K-module defined by
$$(M \otimes N)_{n,m} := \bigoplus_{\substack{a+x=n,\\ b+y=m}} M_{a,b} \otimes N_{x,y},$$
and the induced structure maps.
- The category of bigraded algebras over K has objects bigraded modules $A = \bigoplus_{n,m} A_{n,m}$ over K with a K-linear map $A \otimes A \to A$ making A an algebra with unit. The morphisms are the algebra morphisms which are also morphisms as bigraded K-modules.
- The suspension functors $\Sigma_{a,b} : {}_K\mathrm{Mod} \to {}_K\mathrm{Mod}$ are defined by
$$(\Sigma_{a,b}M)_{m,n} := M_{m-a,n-b}$$
and likewise on functions.
- The graded $\mathrm{Hom}_K^\star$ functor is defined by $\mathrm{Hom}_K^{m,n}(M, N) := {}_K\mathrm{Mod}(M, \Sigma_{m,n}N)$. This makes $\mathrm{Hom}_K^\star$ a bigraded module over K.
- Let A be a bigraded algebra over K. Then we have the notion of bigraded left A-modules. The category of bigraded left A-modules has as objects bigraded K-modules M, together with maps $\mu_M : A \otimes M \to M$, and as morphisms the K-linear maps making the usual diagrams commute. Right A-modules are defined similarly.
- ([ML63, VI.5]) The tensor product of M and N left and right A-modules is defined as the coequalizer in the category of ${}_K\mathrm{Mod}$ of the diagram
$$M \otimes A \otimes N \rightrightarrows M \otimes N \to M \otimes_A N.$$
The arrows are $\mu_M \otimes N$ and $M \otimes \mu_N$. This is isomorphic to the quotient
$$M \otimes N/ \operatorname{im}(\mu_M \otimes N - M \otimes \mu_N).$$

Definition A.1.1. A *Hopf algebroid* is a pair (A, Γ) of bigraded K-algebras with the following structure maps

$$\eta_L : A \to \Gamma, \quad \eta_R : A \to \Gamma,$$

making Γ a bimodule over A, and A-bimodule maps

$\epsilon : \Gamma \to A,$	counit
$\Delta : \Gamma \to \Gamma \otimes_A \Gamma,$	comultiplication
$c : \Gamma \to \Gamma,$	conjugation

satisfying the following relations

$$\begin{aligned} \epsilon\eta_L =& A = \epsilon\eta_R \\ c\eta_L =& c\eta_R \\ cc =& \Gamma \\ (\Gamma \otimes_A \Delta)\Delta =& (\Delta \otimes_A \Gamma)\Delta \\ (\Gamma \otimes_A \epsilon)\Delta =& \Gamma = (\epsilon \otimes_A \Gamma)\Delta \end{aligned}$$

(in the last equation $\Gamma \otimes_A A \cong \Gamma$ is implicit). There exist arrows making the following diagram commutative:

$$\begin{tikzcd} \Gamma & \Gamma \otimes_K \Gamma \arrow[l, "c\cdot\Gamma"'] \arrow[r, "\Gamma\cdot c"] \arrow[d] & \Gamma \\ & \Gamma \otimes_A \Gamma \arrow[ul, dashed] \arrow[ur, dashed] & \\ A \arrow[uu, "\eta_R"] & \Gamma \arrow[l, "\epsilon"] \arrow[r, "\epsilon"] \arrow[u, "\Delta"] & A \arrow[uu, "\eta_L"'] \end{tikzcd}$$

Here $c \cdot \Gamma(\gamma_1 \otimes \gamma_2) = c(\gamma_1)\gamma_2$, and $\Gamma \cdot c(\gamma_1 \otimes \gamma_2) = \gamma_1 c(\gamma_2)$.

Remark A.1.2. This definition can be reformulated in terms of a certain category constructed from (A, Γ) and the structure maps. Let B be a commutative K-algebra. Then we may form a category C_B with objects $\mathrm{Hom}(A, B)$ and morphisms in $\mathrm{Hom}(\Gamma, B)$. Then a morphism is a $h : \Gamma \to B$ such that for $f \xrightarrow{h} g$ the diagram below commutes,

$$\begin{tikzcd} & B & \\ A \arrow[ur, "f"] \arrow[r, "\eta_L"] & \Gamma \arrow[u, "h"] & A \arrow[ul, "g"'] \arrow[l, "\eta_R"'] \end{tikzcd}$$

The identity of $A \xrightarrow{f} B$ is given by $f\epsilon$. The composition of h_1, h_2 is $\mu_B(h_1 \otimes h_2)\Delta$. Then (A, Γ) is a Hopf algebroid if and only if for any B, C_B is a category, which also is a groupoid (i.e., any morphism is an equivalence). The relations on the structure maps then correspond to the axioms of a category and the last diagram to the existence of an inverse to any morphism.

Remark A.1.3. A Hopf algebra is a Hopf algebroid with $\eta := \eta_L = \eta_R$, making Γ a module over A.

Example A.1.4. A common example of a Hopf algebra is the Steenrod algebra in topology. Its dual is also a Hopf algebra. To us, the most important example of a Hopf algebroid is the dual of the motivic Steenrod algebra, Proposition 1.7.3. Other examples of Hopf algebroids are those induced by ring spectra.

Example A.1.5. Let $\Lambda_k(x)$ denote the exterior algebra on one generator x of degree $d > 0$. Then $(k, \Lambda_k(x))$ is a Hopf algebra with the structure maps

$$\begin{aligned}
&\eta : 1 \mapsto 1,\\
&\epsilon : 1 \mapsto 1, x \mapsto 0,\\
&\Delta : 1 \mapsto 1 \otimes 1, x \mapsto x \otimes 1 + 1 \otimes x,\\
&c : x \mapsto -x.
\end{aligned}$$

Definition A.1.6. A left comodule M over a Hopf algebroid (A, Γ) is a left A-module, together with a left A-module map $\psi : M \to \Gamma \otimes_A M$ satisfying

$$\begin{aligned}
&(\Gamma \otimes_A \psi)\psi = (\Delta \otimes_A M)\psi,\\
&(\epsilon \otimes_A M)\psi = M.
\end{aligned}$$

In the last equation $A \otimes_A M \cong M$ is implicit, and $\Gamma \otimes_A M$ is provided with the A-module structure inherited from Γ. The definition of a right comodule is analogous. We also need the notion of Γ-comodule algebras. That is comodules such that the structure map is an algebra map.

Remark A.1.7. If M and N are left Γ-comodules we can form their comodule tensor product, $M \otimes_A N$, with the structure map

$$M \otimes_A N \xrightarrow{\psi_M \otimes \psi_N} \Gamma \otimes_A M \otimes_A \Gamma \otimes_A N \xrightarrow{T} \Gamma \otimes_A \Gamma \otimes_A M \otimes_A N \xrightarrow{\mu_\Gamma} \Gamma \otimes_A M \otimes_A N.$$

Definition A.1.8. An extended Γ-comodule is a comodule of the form $\Gamma \otimes_A M$, for M an A-module, with the structure map $\Delta \otimes M$.

Definition A.1.9. A Hopf algebroid (A, Γ) is connected if the left and right A-modules generated by the degree 0-part of Γ are isomorphic to A.

Proposition A.1.10 ([Rav86, A1.1.3])**.** *If Γ is flat over A (i.e., flat both as a left A-module and as a right A-module), the category of Γ-comodules is an abelian category.*

From now on we assume all our Hopf algebroids to be such that Γ is flat over A. Via η_L we have a canonical left Γ-comodule structure on A, i.e., $\eta_L : A \to \Gamma \cong \Gamma \otimes_A A$, and similarly η_R yields a right Γ-comodule structure.

Definition A.1.11. Let M and N be left Γ-comodules. The set $\mathrm{Hom}_\Gamma(M, N) \subset \mathrm{Hom}_A(M, N)$ consists of the A-linear maps compatible with the comodule structure maps. This is in general only a K-module.

Definition A.1.12 ([Rav86, A1.1.4])**.** The cotensor product of M and N right and left Γ-comodules is defined as the equalizer in ${}_K\mathrm{Mod}$ of the diagram

$$M \,\square_\Gamma\, N \to M \otimes_A N \rightrightarrows \otimes M \otimes_A \Gamma \otimes_A N.$$

The maps are $\psi_M \otimes N$ and $M \otimes \psi_N$. This is isomorphic to $\ker(\psi_M \otimes N - M \otimes \psi_N)$.

Remark A.1.13. From the universal property of the equalizer we have $M \,\square_\Gamma\, \Gamma \cong M$. For a right Γ-comodule L and an extended Γ-comodule $\Gamma \otimes_A M$ we have $L \,\square_\Gamma (\Gamma \otimes_A M) \cong L \otimes_A M$. The isomorphism is via the maps $L \otimes \epsilon \otimes M$ and $\psi_L \otimes M$.

Definition A.1.14 ([Rav86, A1.1.7])**.** A map of Hopf algebroids $f : (A, \Gamma) \to (B, \Sigma)$ is a pair of k-algebra $f_1 : A \to B, f_2 : \Gamma \to \Sigma$ compatible with the structure maps. That is

$$f_1\epsilon = \epsilon f_2, \qquad f_2\eta_L = \eta_L f_1, \qquad f_2\eta_R = \eta_R f_1, \qquad f_2 c = c f_2, \qquad \Delta f_2 = (f_2 \otimes f_2)\Delta.$$

Theorem A.1.15 ([Rav86, A1.1.17])**.** *Let $f : (A, \Gamma) \to (A, \Sigma)$ be a map of connected Hopf algebroids, such that*

1. *$f_2 : \Gamma \to \Sigma$ is surjective.*
2. *$\Gamma \,\square_\Sigma\, A$ is a B-module and a direct summand of Γ via the inclusion.*

Then Γ is isomorphic to $(\Gamma \,\square_\Sigma\, A) \otimes_A \Sigma$ as both a left $\Gamma \,\square_\Sigma\, A$-module and as a right Σ-comodule.

A.2 Homological Algebra on Comodules

We will now proceed to do homological algebra in the category of Γ-comodules. Since Γ is assumed to be a flat A-module, the functor $\Gamma \otimes_A (-)$ preserves injectives, and the category of Γ comodules has enough injectives ([Rav86, A1.2.2]).

Definition A.2.1. We define the right derived functors:

1. $\mathrm{Cotor}^i_\Gamma(M, -)$ is the i-th right derived functor of $M \,\square_\Gamma (-)$.

2. $\mathrm{Ext}^i_\Gamma(M, -)$ is the i-th right derived functor of $\mathrm{Hom}_\Gamma(M, -)$.

We have the following useful lemma:

Lemma A.2.2 ([Rav86, A1.1.6]). *Let M and N be left Γ-comodules with M a projective A-module. Then*

1. $\mathrm{Hom}_A(M, A)$ *is a right* Γ*-comodule.*

2. $\mathrm{Hom}_\Gamma(M, N) = \mathrm{Hom}_A(M, A) \,\square_\Gamma\, N$*, in particular* $\mathrm{Hom}_\Gamma(A, N) = A \,\square_\Gamma\, N$*.*

Remark A.2.3. By the previous lemma $\mathrm{Hom}_\Gamma(A, -) = A \,\square_\Gamma\, -$, hence, $\mathrm{Ext}^i_\Gamma(A, -) = \mathrm{Cotor}^i_\Gamma(A, -)$.

Definition A.2.4. A relative injective module is a direct summand of an extended Γ-comodule (Definition A.1.8). A resolution by injective modules of a comodule M is a long exact sequence

$$M \to R^0 \to R^1 \to \cdots$$

where each R^i is relatively injective, and each map is split over A.

Lemma A.2.5 ([Rav86, A1.2.8]). *If M is Γ-comodule, projective as an A-module, and $\Gamma \otimes_A N$ a relatively injective module. Then*

$$\mathrm{Cotor}^i_\Gamma(M, \Gamma \otimes_A N) = \begin{cases} M \otimes_A N & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

By [Rav86, A1.2.9] relatively injective resolutions can be used to compute Cotor_Γ and Ext_Γ.

A.3 Change of Rings Theorems

Lemma A.3.1 ([Rav86, A1.1.8]). *Let $f : (A, \Gamma) \to (B, \Sigma)$ be a map of Hopf algebroids. Then*

- $\Gamma \otimes_A B$ *is a right* Σ*-comodule.*
- $(\Gamma \otimes_A B) \,\square_\Sigma\, N$ *is a left* Γ*-comodule.*

Proposition A.3.2 ([Rav86, A1.3.12]). *Let $(A, \Gamma) \to (B, \Sigma)$ be a map of connected Hopf algebroids, M a right Γ-comodule, and N a left Σ-comodule, such that*

1. *The map* $\Gamma \otimes_A B \xrightarrow{f_2 \otimes B} \Sigma$ *is surjective.*

2. *The k-module $(\Gamma \otimes_A B) \,\square_\Sigma\, B$ is a B-module, and the canonical inclusion*

$$(\Gamma \otimes_A B) \,\square_\Sigma\, B \hookrightarrow (\Gamma \otimes_A B) \otimes_B B = (\Gamma \otimes_A B),$$

   has a B-linear retraction (the cotensor product is well defined by the previous lemma.)

3. *N is flat as a B-module.*

Then we have an isomorphism

$$\mathrm{Cotor}_\Gamma(M, (\Gamma \otimes_A B) \square_\Sigma N) \cong \mathrm{Cotor}_\Sigma(M \otimes_A B, N).$$

With Remark A.2.3 this specializes to

$$\mathrm{Ext}_\Gamma(A, (\Gamma \otimes_A B) \square_\Sigma N) \cong \mathrm{Ext}_\Sigma(B, N).$$

Corollary A.3.3. *Let $(A, \Gamma) \to (B, \Sigma)$ be a map of Hopf algebroids such that B is finitely generated and free as an A-module and we have the isomorphism $(A, \Gamma) \otimes_A B \cong (B, \Sigma)$. Then*

$$\mathrm{Cotor}_\Gamma(A, A) \otimes_A B \cong \mathrm{Cotor}_\Sigma(B, B)$$

or equivalently

$$\mathrm{Ext}_\Gamma(A, A) \otimes_A B \cong \mathrm{Ext}_\Sigma(B, B).$$

Here $\mathrm{Cotor}_\Gamma(A, A)$ *is an A-module through the map $A = \mathrm{Cotor}_\Gamma(A, \Gamma) \to \mathrm{Cotor}_\Gamma(A, A)$ induced by ϵ (*$\mathrm{Cotor}_\Gamma(A, A)$ *is provided with a ring structure in Equation* (A.2)*).*

Proof. We will apply the proposition above with $M = A$ and $N = B$. We check that the required properties necessary to apply the proposition are present. The first and third properties of Proposition A.3.2 are satisfied, since $\Gamma_1 \otimes_{A_1} A_2 \cong \Gamma_2$ by assumption and B is a flat B-module. The second property is true, since $\Gamma \otimes_A B \cong \Sigma$ we have $(\Gamma \otimes_A B) \square_\Sigma B \cong \Sigma \square_\Sigma B \cong B$ and ϵ is a retraction for $B \hookrightarrow \Sigma$. From the last isomorphism we also have $(\Gamma \otimes_A B) \square_\Sigma N = B$. Hence, Proposition A.3.2 states that

$$\mathrm{Cotor}_\Gamma(A, B) \cong \mathrm{Cotor}_\Sigma(B, B).$$

Since $\mathrm{Cotor}_\Gamma(A, -)$ is an additive functor, and B is a free and finitely generated A-module, we arrive at the result. □

A.4 Cobar Complex

In this section we introduce the cobar resolution of a left Γ-comodule. This is a resolution of relatively injective modules. With this resolution we construct the cobar complex whose homology is Ext. The cobar complex can be given a external product which, under certain hypotheses on the comodules, makes it at differentially graded algebra (Definition A.5.1).

Let $\bar{\Gamma} := \ker \epsilon \cong \operatorname{coker} \eta_L \cong \operatorname{coker} \eta_R$. Since ϵ in general is not a comodule map, $\bar{\Gamma}$ is not in general a Γ-comodule.

Definition A.4.1. Let (A, Γ) be a Hopf algebroid and M a left Γ-comodule. Define the cobar resolution by $D^s_\Gamma(M) := \Gamma \otimes_A \bar{\Gamma}^{\otimes_A n} \otimes_A M$, and differential

$$\begin{aligned} d_s(\gamma_0 \otimes \gamma_1 \otimes \cdots \gamma_s \otimes m) = &\sum_{i=0}^{s} (-1)^i \gamma_0 \otimes \cdots \otimes \gamma_{i-1} \otimes \Delta(\gamma_i) \otimes \gamma_{i+1} \otimes \cdots \otimes m \\ &+ (-1)^{s+1} \gamma_0 \otimes \cdots \otimes \gamma_s \otimes \psi_M(m). \end{aligned}$$

We give each $D^s_\Gamma(M)$ the Γ-comodule structure we get from tensoring with Γ, i.e., we forget the comodule structure on M. The differentials are Γ-comodule morphisms. For a right Γ-comodule L, projective over A, the cobar complex of L and M is $C^*_\Gamma(L, M) := L \square_\Gamma D^*_\Gamma(M)$. Hence,

$$C^s_\Gamma(L, M) = L \otimes_A \bar{\Gamma}^{\otimes_A s} \otimes_A M.$$

An element $a \otimes \gamma_1 \otimes \cdots \otimes \gamma_n \otimes m \in C_\Gamma(L, M)$ will be written as

$$l[\gamma_1|\gamma_2|\cdots|\gamma_n]m$$

or simply $[\gamma_1|\gamma_2|\cdots|\gamma_n]$ if $l = m = 1$.

Proposition A.4.2 ([Rav86, A1.2.12]). *The cobar resolution is a relative injective resolution of M. The homology of the cobar complex is* $\mathrm{Cotor}_\Gamma(L, M)$*, for L a Γ-comodule, projective as an A-module.*

Some facts useful to keep in mind when using the cobar complex are:

- $\eta_L : A \to \Gamma \cong \Gamma \otimes_A A$ is the left Γ-comodule structure on A.
- $\eta_R : A \to \Gamma \cong A \otimes_A \Gamma$ is the right Γ-comodule structure on A.
- The element $a \otimes \gamma \otimes b \in A \otimes_A \Gamma \otimes_A \otimes B$ is equal to $1 \otimes \eta_L(a)\gamma\eta_R(b) \otimes 1$. Similar equalities holds for other tensor products.

To check that the differentials are well defined (i.e., they actually land in $\Gamma \otimes_A \bar{\Gamma}^{\otimes_A s+1} \otimes_A M$) one uses that the tensor product is right exact and check that all the maps of the form

$$\Gamma \otimes \bar{\Gamma} \otimes \cdots \otimes \epsilon \otimes \cdots \otimes \bar{\Gamma} \otimes M$$

vanish on the differentials. This is a straigthforward, but tedious calculation. Using the identification $L \,\square_\Gamma (\Gamma \otimes_A M) \cong L \otimes_A M$ (Remark A.1.13) we obtain an explicit formula for the differentials in the cobar complex

$$\begin{aligned} d_s(l[\gamma_1|\gamma_2|\cdots|\gamma_s]m) = \psi_L(l)\gamma_1|\cdots|\gamma_s]m + \sum_{i=1}^{s}(-1)^i l[\gamma_1|\cdots\gamma_{i-1}|\Delta(\gamma_i)|\gamma_{i+1}|\cdots|\gamma_s]m \\ + (-1)^{s+1} l[\gamma_1|\cdots|\gamma_s|\psi_M(m). \end{aligned} \tag{A.1}$$

Example A.4.3. We will use the cobar resolution to compute $\mathrm{Ext}_{\Lambda_k(x)}(k, k)$, where $\Lambda_k(x)$ is the exterior algebra on one generator x of degree d, with the Hopf algebra structure defined in Example A.1.5. We have $\bar{\Gamma} = k\{x\}$. Then

$$C^s_\Gamma(k) = k \,\square_\Gamma \Gamma \otimes_k \bar{\Gamma}^{\otimes s} \otimes_k k = k\{1 \otimes 1 \otimes x \cdots \otimes x \otimes 1\}.$$

The differentials are

$$\begin{aligned} d_s(1 \otimes 1 \otimes x \cdots \otimes x \otimes 1) = & \, 1 \otimes 1 \otimes 1 \otimes x \cdots \otimes x \otimes 1 \\ & + \sum_{i=1}^{s}(-1)^i 1 \otimes 1 \otimes x \otimes \cdots \otimes (x \otimes 1 + 1 \otimes x)_i \otimes \cdots \otimes x \\ & + (-1)^{s+1} 1 \otimes 1 \otimes x \cdots x \otimes 1 \otimes 1 \\ = & \, 0, \end{aligned}$$

since every term cancels the next one (this can also be seen from degree reasons, since we have $C^s_\Gamma(k) \cong \bar{\Gamma}^{\otimes s} \cong \Sigma_{sd} k$). Hence $\mathrm{Ext}^s_\Gamma(k, k) = \Sigma_{sd} k$ for every $s \geq 0$.

A.4.1 An External Product on Cotor

Using the cobar complex we can provide the derived functors with an exterior product ([Rav86, A1.2.13]). Let I_1^* and I_2^* be relative injective resolutions of N_1 and N_2, respectively. Then $I_1^* \otimes_A I_2^*$ is a realtive injective resolution of $N_1 \otimes_A N_2$. There is a canonical map

$$\mathrm{Cotor}_\Gamma(M_1, N_1) \otimes \mathrm{Cotor}_\Gamma(M_1, N_1) \to H((M_1 \,\square_\Gamma P_1^*) \otimes (M_2 \,\square_\Gamma P_2^*))$$

The map

$$(M_1 \otimes_A P_1^*) \otimes (M_2 \otimes_A P_2^*) \to (M_1 \otimes_A M_2) \otimes_A (P_1^* \otimes_A P_2^*)$$

induces a map

$$(M_1 \,\square_\Gamma P_1^*) \otimes (M_2 \,\square_\Gamma P_2^*) \to (M_1 \otimes_A M_2) \,\square_\Gamma (P_1^* \otimes_A P_2^*).$$

Taking homology we obtain a pairing

$$\mathrm{Cotor}_\Gamma(M_1, N_1) \otimes \mathrm{Cotor}_\Gamma(M_2, N_2) \to \mathrm{Cotor}_\Gamma(M_1 \otimes_A M_2, N_1 \otimes_A N_2). \tag{A.2}$$

If $M_1 = M_2 = M$ and $N_1 = N_2 = N$ are Γ-comodule algebras composition with $\mathrm{Cotor}_\Gamma(\mu_M, \mu_N)$ make $\mathrm{Cotor}_\Gamma(M, N)$ a graded commutative, associative algebra. If we specialize to $M = N = A$ we have obtained an algebra structure on $\mathrm{Ext}_\Gamma(A, A)$. It is possible to give the cobar complex a pairing which induces the pairing in Equation (A.2). This is done in [Rav86, A1.2.15]. If $M_1 = M_2 = N_1 = N_2 = A$ this makes $C_\Gamma(A, A)$ a differentially graded algebra, with differential of degree 1. Furthermore this makes $\mathrm{Ext}_\Gamma(A, A)$ an associative and graded commutative algebra.

A.5 Massey Products

Definition A.5.1 ([Wei94, 4.5.2]). A differentially graded algebra is a pair (A, d) of an algebra A and a differential $d : A \to A$, which satisfy the Leibniz rule. We restrict to the cases when A is an algebra over $\mathbb{Z}/2$, and the degree of d is 1.

When (A, d) is a differentially graded algebra the homology of A is a graded algebra. The canonical example of a differentially graded algera is the cobar complex, $C_\Gamma(A, A)$ for a Hopf algebroid (A, Γ).

Let $\alpha_1, \alpha_2, \alpha_3$ be classes in $H(A)$ of degrees s_1, s_2, s_3, represented by cocycles $a_1, a_2, a_3 \in A$, such that

$$\alpha_1\alpha_2 = 0 = \alpha_2\alpha_3.$$

Then there exist $u_1, u_2 \in A$ such that $d(u_1) = a_1a_2$ and $d(u_2) = a_2a_3$. The Massey product of $\alpha_1, \alpha_2, \alpha_3$ is then defined as

$$\langle\alpha_1, \alpha_2, \alpha_3\rangle := [a_1u_2 + u_1a_3].$$

Here $[x]$ denotes the class of $x \in A$ in $H(A)$. This is not well defined, since u_1 and u_2 are only defined up to addition of a cocyle. Hence, the Massey product is only defined up to elements in $\alpha_1 H^{s_2+s_3-1}(A) \oplus \alpha_3 H^{s_1+s_2-1}(A)$. This group is denoted by $\mathrm{In}\langle\alpha_1, \alpha_2, \alpha_3\rangle$. It is often zero, in which case the Massey product is well defined. Otherwise, the Massey product is the coset $[u_1a_3 + u_2a_1] + \mathrm{In}\langle\alpha_1, \alpha_2, \alpha_3\rangle$. It is possible to define higher order Massey products, however, we do not need these.

The following lemma is one of several "juggling"-theorems, which give rules for manipulating Massey products.

Lemma A.5.2 ([Rav86, A1.4.6]). *Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be elements as above, such that $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ and $\langle\alpha_2, \alpha_3, \alpha_4\rangle$ are defined. Then*

$$\alpha_1\langle\alpha_2, \alpha_3, \alpha_4\rangle = \langle\alpha_1\alpha_2, \alpha_3\rangle\alpha_4.$$

B Some Number Theory

Large parts of motivic homotopy theory require the ground field to be perfect.There are several equivalent definitions of perfect fields. For fields in positive characteristic one definition is to require that the Frobenius map is an automorphism. In particular finite fields are perfect, since the group of units are p-divisible in $\mathbb{F}_{p^n}$ [Isa94, p. 297].

Lemma B.1.1. *For any finite field $\mathbb{F}_q$, the group of units $\mathbb{F}_q^\times$ is cyclic.*

Proof. A group G is cyclic if and only if the group exponent $d := \mathrm{lcm}_{x \in G}(\mathrm{ord}(x)) = |G|$. We always have $d \leq |G|$. The equation $x^d = 1$ has $q - 1 \leq d$ solutions. Hence, $d = q - 1 = |\mathbb{F}_q^\times|$ and the group is cyclic. $\square$

Proposition B.1.2. *The following sequence is exact for any finite field*

$$1 \longrightarrow \mu_2 \hookrightarrow \mathbb{F}_q^\times \xrightarrow{x \mapsto x^2} \mathbb{F}_q^\times \xrightarrow{x \mapsto x^{(q-1)/2}} \mu_2 \longrightarrow 1.$$

Proof. Exactness at the first group is evident. Exactness at the first $\mathbb{F}_q^\times$ follows from the fact that $x^2 - 1 = (x-1)(x+1)$. It is exact at $\mathbb{F}_q^\times$, since if g is the generator and $x = g^k$ is such that $x^{(q-1)/2} = 1 = g^{k(q-1)/2}$, then $2|k$, hence x is a square. $\square$

Lemma B.1.3. *Let $q = p^m$ for some prime p. For ν_2, the 2-adic valution, we have*

$$\nu_2(q^n - 1) = \begin{cases} \nu_2(q-1) + \nu_2(n) & q \equiv 1 \ (4), \\ \nu_2(q+1) + \nu_2(n) & q \equiv 3 \ (4), \, n \text{ even}, \\ 1 & q \equiv 3 \ (4), \, n \text{ odd}. \end{cases}$$

Proof. Use that ν_2 is multiplicative, the familiar identity $(q^n - 1)/(q-1) = \sum_{i=0}^{n-1} q^i$ and induction. $\square$

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